Pre Calculus

Content:

Prerequisite Algebra Review

1 The Set of Real Numbers

Informally, a real number is a number with a decimal representation, and we represent the set of real numbers by an $\mathbb{R}$. Now, there are different subsets of real numbers.

The first subset to consider is what we call the natural numbers, and these are just the counting numbers that we are used to: 1 2, 3, 4, and so on. We represent the set of natural numbers by an $\mathbb{N}$.

$$\mathbb{N} = \{1, 2, 3, 4 ...\}$$

The second subset to consider is what we call the integers. The integers are all these natural numbers together with their negatives and zero, and we represent a set of integers by a $\mathbb{Z}$. Therefore, $\mathbb{Z}$ is all the negative natural numbers together with 0, and then all these natural numbers.

$$\mathbb{Z} = [...-3, -2, -1, 0, 1, 2, 3 ...\}$$

The third subset to consider is what we call the rational numbers, and we represent the set of rational numbers by a $\mathbb{Q}$. These are fractions, or ratios of integers.

$$\frac{a}{b}, \text{ b} \neq 0, \text{ a and b are integers}$$

Now, the decimal representation of a rational number either terminates or repeats.

Example:

$$\frac{1}{2} = 0.5 \qquad \text{terminates}$$
$$\frac{1}{3} = 0.3\overline{3}... \qquad \text{repeats}$$

If a real number, represented by an $\mathbb{R}$, is not rational, then it is what we call an irrational number. The set of rational numbers we represent by capital I, therefore, these are real numbers that are not rational.

A real number is either rational or irrational. Let us write that up here.

$$\mathbb{N} \subset^{1} \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Rational numbers together with the irrational numbers, and we have that the natural numbers are contained in the integers, contained in the rationals, and are contained in the reals.

Example:

$$S = \left\{-\sqrt{5}, -1, \pi\pi\frac{3}{4}, 3, \sqrt{7}, 10, -\sqrt{4}, 0, \sqrt{\frac{81}{4}}\right\}$$

These will be our natural numbers: $\{3, 10\}$.

The integers are these natural numbers together with 0: $\left\{3, 10, 0, -1, \left(-\sqrt{4}\right) = -2\right)\right\}$.

What about the rationals? $\left\{3, 10, 0, -1, \left(-\sqrt{4}\right) = -2\right), \frac{3}{4}, \left(\sqrt{\frac{81}{4}} = \frac{\sqrt{81}}{\sqrt{4}} = \frac{9}{2}\right)\right\}$.

All right, finally the irrational numbers: $\left\{-\sqrt{5}, \pi, \sqrt{7}\right\}$

[1] $\subset$ means contained

2 Properties of Real Numbers

For example, which real number property justifies each of these 4 statements?

$5*(ab)=(5*a)*b$	Associative property of multiplication
$4+r=r+4$	Commutative property of addition
$3*u+7*u=(3+7)u$	Distributive property
$(m+n)*0=0$	Multiplication property of zero

3 Scientific Notation

Scientific notation is a way to express, or rewrite, a large number or a small number in a concise way. For example, let us write

569,000

in scientific notation. To write a number in scientific notation we can rewrite it in the form $a * 10^n$, where a, in absolute value, is between 1 and 10, and n is an integer.

The first thing we need to do here with our number is to move the decimal point, until we get a number whose absolute value is between 1 and 10. You might be thinking, there is no decimal point there. But is not

569,000 = 569,000.00 ?

Therefore, there is a decimal point.

If we move this decimal point, 1, 2, 3, 4, 5 places to the left, we get 5.69, which is equal to our a. It is the number we are looking for here, whose absolute value is between 1 and 10. Now, what about n? We need to find an integer n such that $5.69*10^{\mathbf{n}}$ is equal to this 569,000.

Well, since we moved the decimal point 5 units to the left, we would have to multiply by 10^5, or $n = 5$:

$$5{,}69 * 10^5 = 569{,}000$$

Therefore, our answer is $5.69*10^5$.

What about the other way? In other words, what about going from a small number to scientific notation. Again, we will need to find our a, and our n, where a is a number in absolute value between 1 and 10, and n is an integer.

Let us look at our number here:

-0.000,000,691

In order to find our a, we need to move this decimal point 7 places to the right. Therefore $a = 6.91$. However, what is about our n? We need to find an integer n, such that $-6.91 * 10^n$ is equal to -0.000,000,691.

Since we moved our decimal point 7 units to the right to get to a, we need to multiply by n = -7.

$$-6.91 * 10^{-7} = -0.000{,}000{,}691$$

Therefore, our answer then is $-6.91 * 10^{-7}$.

4 Properties of Integer Exponents

For n and m integers and a and b real numbers, we have the following properties.

$a^n * a^m = a^{n+m}$	Product Rule	$a^2 * a^3 = a^{2+3} = a^5$	
$(a^n)^m = a^{n*m}$	Power of a power rule	$(a^2)^3 = a^{2*3} = a^6$	
$(ab)^m = a^m * b^m$	Power of a product rule	$(ab)^2 = a^2 * b^2$	
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	Power of a quotient rule	$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$	with $b \neq 0$
$\frac{a^m}{b^n} = a^{m-n}$	Quotient rule	$\frac{a^5}{a^2} = a^{5-2} = a^3$	
$a^0 = 1$	Zero exponent rule	$3^0 = 1$	with $a \neq 0$
$a^{-n} = \frac{1}{a^n}$	Negative exponent rule	$a^{-3} = \frac{1}{a^3}$	

Simplify the following expressions, and write your answer using only positive exponents

$$
\begin{aligned}
&\left(2*w^{-5}*v^{-6}\right)*\left(3*u^{7}*u^{2}\right)*\left(5*v^{7}*w^{4}\right)\\
&=\left(2*3*5\right)*\left(w^{-5}*w^{4}\right)*\left(v^{-6}*v^{7}\right)*\left(u^{7}*u^{2}\right)\\
&=30*w^{-5+4}*v^{-6+7}*u^{7+2}\\
&=30*w^{-1}*v^{1}*u^{9}\\
&=\underline{\frac{30vu^{9}}{w}}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{6*m*n^{-2}}{3*n^{2}*m^{-1}}\right)^{-3}\\
&=\left(\frac{6}{3}*\frac{m}{m^{-1}}*\frac{n^{-2}}{n^{2}}\right)^{-3}\\
&=\left(2*m^{1-(-1)}*n^{-2-2}\right)^{-3}\\
&=\left(2*m^{2}*n^{-4}\right)^{-3}\\
&=2^{-3}*\left(m^{2}\right)^{-3}*\left(n^{-4}\right)^{-3}\\
&=\frac{1}{2^{3}}*m^{-6}*n^{12}\\
&=\underline{\frac{n^{12}}{8*m^{6}}}
\end{aligned}
$$

5 Adding and Subtracting Polynomials

For example, let us add two polynomials together. The first thing we will do is remove these parentheses. In other words, this is:

$$
\begin{aligned}
&\left(6x^{2}-x-2\right)+\left(7x^{2}-5x+3\right)\\
&=6x^{2}-x-2+7x^{2}-5x+3
\end{aligned}.
$$

Now, let us focus on grouping like terms. The 6 * x^2 and 7 * x^2 are like terms, as well as the -x and -5x, as are -2 and 3. Therefore, grouping these together gives us:

$$\underline{=13x^{2}-6x+1}$$

Now, sometimes you will see polynomials added vertically. We have this first polynomial $6x^2$ -x - 2, and then we are adding to this the second polynomial $7x^2$ - 5x + 3. Now the like terms are aligned vertically. Therefore, adding them will give us the same answer.

$$
\begin{array}{r}
6x^{2}-x-2\\
+\ \ 7x^{2}-5x+3\\
\hline
\underline{=13x^{2}-6x+1}
\end{array}
$$

Let us look at another example.

$$\left(-u^{2}+5u\right)-\left(3u^{2}+7u+6\right)$$

Now a subtraction. We have to be very careful when we remove our parentheses and distribute this -, or -1, to all three of these terms in the second polynomial.

$$
\begin{aligned}
&\left(-u^{2}+5u\right)-\left(3u^{2}+7u+6\right)\\
&=-u^{2}+5u-3u^{2}-7u-6
\end{aligned}
$$

Again, we will group like terms, the $-u^2$ and $-3u^2$ are like terms, as well as the 5u and -7u. Therefore, grouping them together gives us:

$$
\begin{aligned}
&=\left(-u^{2}-3u^{2}\right)+\left(5u-7u\right)-6\\
&\underline{=-4u^{2}-2u-6}
\end{aligned}
$$

Now, if we try to do this vertically, we have to be careful here, because of the subtraction. In other words, we have $-u^2 + 5u$, and then -, but it is - the entire second polynomial, this $-(3u^2 + 7u + 6)$.

$$\begin{array}{r} -u^2+5u \\ \underline{-\left(3u^2+7u+6\right)} \end{array}$$

Now we have to be very careful with this constant term in the first polynomial. Let us put a +0, because when we are subtracting, it is really 0 – 6, or -6, which is the same answer.

$$\begin{array}{r} -u^2+5u+0 \\ \underline{-\left(3u^2+7u+6\right)} \\ \underline{-4u^2-2u-6} \end{array}$$

Often, when we subtract vertically, you will see the following instead. We have the $-u^2 + 5u$, but then subtraction means we add the opposite. Therefore, + and then the opposite of the second polynomial is $-3^2 - 7u - 6$. Now we can add, we get:

$$\begin{array}{r} -u^2+5u+0 \\ \underline{+-3u^2+7u+6} \\ \underline{-\ 4u^2-2u-6} \end{array}$$

This again is the same answer. Therefore,, be careful with subtraction.

All right, let us look at one more example. Notice now, we have three polynomials, but we are subtracting the second, and adding the third. We need to be careful when we are removing our parentheses.

$$\begin{aligned} &\left(-5w^2-5w+3\right)-\left(-4w^2-2w-2\right)+\left(3w-2w^2+1\right) \\ &=-5w^2-5w+3+4w^2+2w+2+3w-2w^2+1 \\ &=\left(-5w^2+4w^2-2w^2\right)+\left(-5w+2w+3w\right)+\left(3+2+1\right) \\ &\underline{=-3w^2+6} \end{aligned}$$

If we wanted to do this vertically, we need to be careful here for a few different reasons. We have this first polynomial, $-5w^2 - 5w + 3$, but then, looking back up there, we are subtracting this entire second polynomial, which is equivalent to adding its opposite, and its opposite is $4w^2 + 2w + 2$. Then we are adding this third polynomial. However, this third polynomial is not written in order. We want to put this $-2w^2$ first that we have $-2w^2 + 3w + 1$, and then we are going to add.

$$\begin{array}{l} -5w^2-5w+3 \\ +4w^2+2w+2 \\ \underline{-2w^2+3w+1} \\ \underline{-3w^2+6} \end{array}$$

6 Multiplication of Binomials

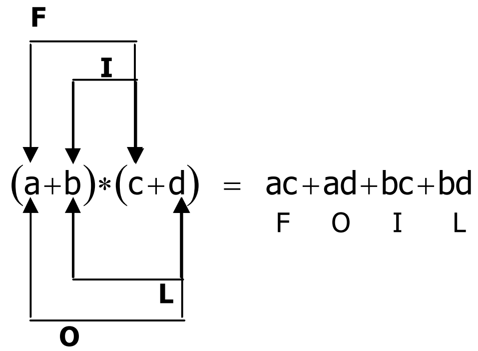
For example:

$$\begin{aligned} &\left(2x+5y\right)*\left(3x-y\right) && \text{Extend} \\ &=\left(2x*3x\right)+\left(5y*3x\right)+\left(2x*-y\right)+\left(5y*-y\right) && \text{Multiply} \\ &=6x^2+15xy-2xy-5y^2 && \text{Simplify} \\ &\underline{=6x^2+13xy-5y^2} \end{aligned}$$

Now this type of multiplication comes up often. There is an acronym used to describe it, and the acronym is 'FOIL'.

The '*F*' in 'FOIL' stands for *first*, which means we multiply the first terms in each binomial. The '*O*' stands for *outer*, which means we multiple the outer terms of the two binomials. The '*I*' stands for *inner*, which means we multiply the inner terms of the two binomials. Finally, the '*L*' stands for *last*, which means we multiply the last terms of the two binomials. The answer to this multiplication is the sum of all of these.

F + O + I + L



Let us apply this method here to see that we get the same answer that we just found.

$$
\begin{aligned}
&(2x+5y)*(3x-y)\\
F &= 2x*3x = 6x^2\\
O &= 2x*-y = -2xy\\
I &= 5y*3x = 15yx = 15xy\\
L &= 5y*-y = -5y^2\\
&= 6x^2 - 2xy + 15xy - 5y^2\\
&\underline{= 6x^2 + 13xy - 5y^2}
\end{aligned}
$$

This gives us the same answer. 'FOIL' is a quick way to do distributive multiplication of two binomials.

Let us see another example. Let us multiply these two binomials here. Well, we can apply the 'FOIL'-method we just saw.

$$
\begin{aligned}
(2x-1)*(2x+1) &= (2x*2x)+(2x*1)+(2x*-1)+(1*-1)\\
&\underline{= 4x^2 - 1}
\end{aligned}
$$

Now it should be pointed out that these binomials are special, and that we are multiplying together the difference and sum of the same two terms. We have 2x and 2x, and 1 and 1. There is a special formula in this type of case, and the formula is, that (A – B) * (A + B) = A^2 - B^2, because the outer and inner terms will always cancel, which is what we just saw. That is, in our case, our A = 2x and our B = 1.

Let us do another example. Now be careful here with this power of 2.

$$
\begin{aligned}
(3y-5)^2 &= (3y-5)*(3y-5)\\
&= (3y*3y)+(3y*-5)+(-5*3y)+(-5*-5)\\
&= 9y^2 - 15y - 15y + 25\\
&\underline{= 9y^2 - 30y + 25}
\end{aligned}
$$

Now again, this is the common type of multiplication, where we are multiplying a binomial by itself, and there is a special formula again in this type of case. The formula is that $(A - B)^2 = A^2$ -2AB + B^2. That is the outer and inner terms are the same, therefore, there will be two of them, which we just saw with A = 3y and b = 5.

7 Multiplication of Polynomials

The first thing we do when we multiply polynomials is, we use the distributive property, and then we combine like terms when necessary.

$$(2x^2)*(3x^2 - x + 4)$$

The first thing we will do is distribute this monomial to each of the 3 terms in the trinomial, which gives us:

$$
\begin{aligned}
(2x^2)*(3x^2 - x + 4) &= (2x^2*3x^2)+(2x^2*-x)+(2x^2*4)\\
&\underline{{}' = 6x^4 - 2x^3 + 8x^2}
\end{aligned}
$$

All right, let us look at another example. Let us multiply this binomial and this trinomial.

$$\left(x^2+1\right)*\left(2x^2-3x+4\right)$$

The first thing we will do is distribute the entire binomial to each of these three terms in the trinomial.

$$\begin{aligned}\left(x^2+1\right)*\left(2x^2-3x+4\right) &= \left(x^2+1*2x^2\right)+\left(x^2+1*-3x\right)+\left(x^2+1*4\right)\\ &= \left(x^2*2x^2\right)+\left(1*2x^2\right)+\left(x^2*-3x\right)+\left(1*-3x\right)+\left(x^2*4\right)+\left(1*4\right)\\ &= 2x^4+2x^2-3x^3-3x+4x^2+4\\ &= \underline{2x^4-3x^3+6x^2-3x+4}\end{aligned}$$

8 Multiple Operations with Polynomials

Let us perform the indicator operations and simplify.

$$\left(x^3-2\right)-\left[2x^3-\left(3x+4\right)\right]$$

The first thing we will do is distribute the -, which really means -1 to each of these two terms here, that is:

$$x^3-2-2x^3+3x+4$$

Let us combine like terms.

$$-x^3+3x+2$$

Let us look at another example. Let us perform the indicated operations and simplify.

$$\underbrace{\left(3u-2v\right)*\left(3u+2v\right)}-\left(2u-3v\right)^2$$

Well we can begin by foiling this product out.

$$9u^2+6uv-6uv-4v^2-\left(2u-3v\right)^2$$

Now what is this here? There is a common mistake that students make often that should be pointed out. They want to apply this power of 2 to both of these two terms. We cannot do that, because this is not equal to:

$$\left(2u\right)^2+\left(3v\right)^2$$

Therefore, we need to foil again.

$$\begin{aligned}&=\left(2u-3v\right)*\left(2u-3v\right)\\ &=4u^2-6uv-6uv+9v^2\end{aligned}$$

Substitute this into the original equation:

$$9u^2+6uv-6uv-4v^2-\left(4u^2-6uv-6uv+9v^2\right)$$

Then distribute the -, or -1, that gives us:

$$9u^2+6uv-6uv-4v^2-4u^2+6uv+6uv-9v^2$$

Now 6uv will cancel, and then we can combine the rest of the like terms.

$$5u^2+12uv-13v^2$$

All right, let us look at one more example. Let us perform the indicated operations and simplify.

$$3*\boxed{\left(x+h\right)^2}-5*\left(x+h\right)+7-\left(3x^2-5x+7\right)$$

Again, be very careful not to apply this power of 2 to both of these terms. We need to foil (x + h) * (x + h). This is:

$$3*\left(x^2+2xh+h^2\right)-5*(x+h)+7-\left(3x^2-5x+7\right)$$

Now we will distribute the 3 to these three terms, the -5 to these two terms as well as the -1 to these three terms, which gets us:

$$3x^2+6xh+3h^2-5x-5h+7-3x^2+5x-7$$

After simplifying our equation looks like this:

$$\underline{6xh+3h^2-5h}$$

9 Rational Exponents

For *m* and *n*, natural numbers, and *b* any real number, we have the following.

$$b^{m/n}=\begin{cases}\left(b^{1/n}\right)^m\\ \left(b^m\right)^{1/n}\end{cases}\qquad \text{b can't be -, when n is even}$$

Let us see an example. Let us say we want to compute

$$8^{2/3}$$

If we use the first method, this is equal to

$$\begin{aligned}8^{2/3}&=\left(8^{1/3}\right)^2\\&=(2)^2\\&\underline{=4}\end{aligned}$$

If we use the second method, we would first square

$$\begin{aligned}8^{2/3}&=\left(8^2\right)^{1/3}\\&=(64)^{1/3}\\&\underline{=4}\end{aligned}$$

It does not matter. They are the same. You can use either method, but in some cases we want to use one method over the other.

For example, let us say we wanted to compute

$$\begin{aligned}27^{4/3}&=\left(27^{1/3}\right)^4\\&=(3)^4\\&\underline{=81}\end{aligned}$$

We most definitely would want to do it this first way, but if we try to use a second method, we would have to raise 27 to the fourth power, and then take the cube root of that, it is much more difficult, right?

What about rational exponents, which are -. For *m* and *n* are natural numbers, and *b* is any real number, we have:

$$b^{-(m/n)}=\frac{1}{b^{m/n}}\qquad \text{b can't be -, when n is even}$$

For example:

$$9^{-\left(3/2\right)} = \frac{1}{9^{\left(3/2\right)}}$$
$$= \frac{1}{\left(9^3\right)^{1/2}} \quad or \quad \frac{1}{\left(9^{1/2}\right)^3}$$

Now, the question is which method we would use to compute this denominator. Would we compute 9^3 raised to the ½ power, or 9 to the ½ whole thing cubed. This here is much more promising, is it not?

Because otherwise we would have to compute 9 cubed.

$$\frac{1}{\left(9^{1/2}\right)^3} = \frac{1}{(3)^3}$$
$$= \frac{1}{27}$$

10 Simplified Radical Form

Let us discuss the simplified radical form.

In order for an algebraic expression to be in simplified radical form, all of the following must be true.

- The first property that must hold is that no radicand contains a factor to a power greater than or equal to the index of the radical.

For example:

$$\sqrt[3]{y^5}$$

Is not simplified, because the power of the factor y is greater than the index of the radical (5 > 3).

- The second property that must hold is that no power of the radicand and the index of the radical have a common factor other than 1.

For example:

$$\sqrt[9]{x^{12}}$$

Is not be simplified, because 9 and 12 have a common factor of 3.

- The third property that needs to hold is that no radical appears in the denominator.

For example:

$$\frac{2}{\sqrt{7}}$$

Is not simplified, because we have the √7 in the denominator.

- The last property that must hold is that no fraction appears within a radical.

For example:

$$\sqrt{\frac{5}{4}}$$

Is not simplified, because we have this 5/4 within the radical.

All right, let us see an example of how we do put an algebraic break expression into simplified radical form. Let us put this expression into simplified radical form, and we are assuming here that *x* and *y* represent positive real numbers.

$$\sqrt[4]{16 * x^6 * y^9}$$

The first thing we should notice here is that the radicand contains factors raised to powers greater than the index of 4. We have the 6, as well as the 9. Since we are simplifying a fourth root, we need to focus on the perfect fourth power factors of the radicand. Now, something has a perfect fourth power factor when its exponent is a multiple of 4. We extract the perfect fourth power factors here. Therefore, we rewrite x^6 as $x^4 * x^2$, and we are going to rewrite y^9 as $y^8 * y$, and this is:

$$\sqrt[4]{16 * x^6 * y^9} = \sqrt[4]{2^4 * x^4 * x^2 * y^8 * y}$$

Then we rewrite y^8 as $(y^2)^4$, and then * y.

$$= \sqrt[4]{2^4 * x^4 * x^2 * \left(y^2\right)^4 * y}$$

Now, let us group together all perfect fourth power factors, namely 2^4, x^4, and $(y^2)^4$.

$$= \sqrt[4]{2^4 * x^4 * \left(y^2\right)^4 * x^2 * y}$$

Then, by properties of exponents, this is:

$$= \sqrt[4]{\left(2 * x * y^2\right)^4 * x^2 * y}$$

Again, by properties of exponents, this is:

$$= \sqrt[4]{\left(2 * x * y^2\right)^4} * \sqrt[4]{* x^2 * y}$$

Now this is:

$$= 2xy^2 * \sqrt[4]{x^2 y}$$

The question is, are we done? We are not, because 2 and 4 have a common factor other than 1. Therefore, let us first split this up as $2xy^2$, and then the $\sqrt[4]{x^2}$, and then the $\sqrt[4]{y}$.

$$= 2xy^2 * \sqrt[4]{x^2} * \sqrt[4]{y}$$

Now let us convert the term $\sqrt[4]{x^2}$ to rational exponent form. In other words, this is $x^{2/4}$, which is $x^{1/2}$, or $\sqrt{x}$. This is equal to:

$$\underline{= 2xy^2 * \sqrt{x} * \sqrt[4]{y}}$$

Which would be in simplified radical form.

All together looks like this.

$$\begin{aligned}
\sqrt[4]{16 * x^6 * y^9} &= \sqrt[4]{2^4 * x^4 * x^2 * y^8 * y} \\
&= \sqrt[4]{2^4 * x^4 * x^2 * \left(y^2\right)^4 * y} \\
&= \sqrt[4]{2^4 * x^4 * \left(y^2\right)^4 * x^2 * y} \\
&= \sqrt[4]{\left(2 * x * y^2\right)^4 * x^2 * y} \\
&= \sqrt[4]{\left(2 * x * y^2\right)^4} * \sqrt[4]{* x^2 * y} \\
&= 2xy^2 * \sqrt[4]{x^2} * \sqrt[4]{y} \\
&\underline{= 2xy^2 \sqrt{x} \sqrt[4]{y}}
\end{aligned}$$

11 Square Root Addition and Subtraction

For example, let us add

$$5 * \sqrt{45} + \sqrt{20}$$

The thing we will have to do is, we will simplify each of the square root terms by extracting all perfect squares. In other words, this is:

$$\begin{aligned} 5 * \sqrt{45} + \sqrt{20} &= 5 * \sqrt{9 * 5} + \sqrt{4 * 5} \\ &= 5 * \sqrt{9} * \sqrt{5} + \sqrt{4} * \sqrt{5} \\ &= 5 * 3 * \sqrt{5} + 2 * \sqrt{5} \\ &= 15 * \sqrt{5} + 2 * \sqrt{5} \\ &\underline{= 17 * \sqrt{5}} \end{aligned}$$

All right, let us look at another example. Let us simplify this.

$$\sqrt{27} + 3 * \sqrt{25 * 3} - 2 * \sqrt{12}$$

Again, let us start off by simplifying each square root term by extracting all perfect squares, and combining the results. In other words:

$$\begin{aligned} \sqrt{27} + 3 * \sqrt{25 * 3} - 2 * \sqrt{12} &= \sqrt{9 * 3} + 3 * \sqrt{25} * \sqrt{3} - 2 * \sqrt{4 * 3} \\ &= \sqrt{9} * \sqrt{3} + 3 * \sqrt{25} * \sqrt{3} - 2 * \sqrt{4} * \sqrt{3} \\ &= 3 * \sqrt{3} + 3 * 5 * \sqrt{3} - 2 * 2 * \sqrt{3} \\ &= 3 * \sqrt{3} + 15 * \sqrt{3} - 4 * \sqrt{3} \\ &\underline{= 14 * \sqrt{3}} \end{aligned}$$

Let us see one more.

$$5 * \sqrt{24} - 4 * \sqrt{32} + 3 * \sqrt{18} - 2 * \sqrt{54}$$

Again, we will start by working with each square root separately, extracting the perfect squares and combine terms with common radicals. In other words, this is:

$$\begin{aligned} 5 * \sqrt{24} - 4 * \sqrt{32} + 3 * \sqrt{18} - 2 * \sqrt{54} &= 5 * \sqrt{4 * 6} - 4 * \sqrt{16 * 2} + 3 * \sqrt{9 * 2} - 2 * \sqrt{9 * 6} \\ &= 5 * \sqrt{4} * \sqrt{6} - 4 * \sqrt{16} * \sqrt{2} + 3 * \sqrt{9} * \sqrt{2} - 2 * \sqrt{9} * \sqrt{6} \\ &= 5 * 2 * \sqrt{6} - 4 * 4 * \sqrt{2} + 3 * 3 * \sqrt{2} - 2 * 3 * \sqrt{6} \\ &= 10 * \sqrt{6} - 16 * \sqrt{2} + 9 * \sqrt{2} - 6 * \sqrt{6} \\ &\underline{= 28 * \sqrt{6} * \sqrt{2}} \end{aligned}$$

12 Multiplication involving Square Roots

For example, let us simplify this expression here.

$$4 * \sqrt{20} * \sqrt{80}$$

Let us begin by first simplifying these square roots separately. Therefore, what is the square root of 20? Well, let us extract all perfect squares here. In other words, 20 is 4 * 5, and then by properties of radicals, this is equal to the √4 * √5, which is = 2 * √5.

$$\begin{aligned} \sqrt{20} &= \sqrt{4 * 5} \\ &= \sqrt{4} * \sqrt{5} \\ &= 2 * \sqrt{5} \end{aligned}$$

What about the √80? This is = √(16 * 5), which is equal to, again by properties of radicals, √16 * √5, or 4 * √5.

$$\begin{aligned}\sqrt{80} &= \sqrt{16 * 5} \\ &= \sqrt{16} * \sqrt{5} \\ &= 4 * \sqrt{5}\end{aligned}$$

Let us put these simplifications back into our equation, which gives us:

$$\begin{aligned}4 * \sqrt{20} * \sqrt{80} &= 4 * \left(2 * \sqrt{5}\right) * \left(4 * \sqrt{5}\right) \\ &= \left(4 * 2 * 4\right) * \left(\sqrt{5} * \sqrt{5}\right) \\ &= 32 * (5) \\ &= 160\end{aligned}$$

Now we could have begun a bit different, and just applied this property from the very beginning. In other words this is:

$$\begin{aligned}4 * \sqrt{20} * \sqrt{80} &= 4 * \sqrt{20 * 80} \\ &= 4 * \sqrt{1{,}600}\end{aligned}$$

However, 1,600 is a pretty big number, therefore, we decided to simplify the smaller radicals first.

All right, let us look at another example. Let us simplify this.

$$\left(5 * \sqrt{3} + 4 * \sqrt{2}\right) * \left(3 * \sqrt{3} - 2 * \sqrt{2}\right)$$

What we do is we begin by foiling, and combining the numbers and terms.

$$\begin{aligned}&= \left(5 * \sqrt{3}\right) * \left(3 * \sqrt{3}\right) + \left(5 * \sqrt{3}\right) * \left(-2 * \sqrt{2}\right) + \left(4 * \sqrt{2}\right) * \left(3 * \sqrt{3}\right) + \left(4 * \sqrt{2}\right) * \left(-2 * \sqrt{2}\right) \\ &= 15 * \sqrt{9} - 10 * \sqrt{6} + 12 * \sqrt{6} - 8\sqrt{4} \\ &= 15 * 3 + 2 * \sqrt{6} - 8 * 2 \\ &= 45 + 2 * \sqrt{6} - 16 \\ &= 29 + 2 * \sqrt{6}\end{aligned}$$

13 Rationalizing a Denominator

For example, let us rationalize the denominator of this expression, and write our answer in simplified form.

$$\frac{3 * \sqrt{2} - 2 * \sqrt{5}}{3 * \sqrt{5} - 4 * \sqrt{2}}$$

Looking at this denominator here, we have two irrational numbers.

Remember to rationalize the denominator means to make the denominator a rational number. We can do that by multiplying both, the numerator and denominator, by the conjugate of the denominator. Remember the conjugate is the same expression, but we just change the - to a +.

Let us put parentheses here, so that we know that we are going to foil both the numerator and denominator. What do we get?

$$\begin{aligned}\frac{3 * \sqrt{2} - 2 * \sqrt{5}}{3 * \sqrt{5} - 4 * \sqrt{2}} &= \frac{\left(3 * \sqrt{2} - 2 * \sqrt{5}\right)}{\left(3 * \sqrt{5} - 4 * \sqrt{2}\right)} * \frac{\left(3 * \sqrt{5} - 4 * \sqrt{2}\right)}{\left(3 * \sqrt{5} - 4 * \sqrt{2}\right)} \\ &= \frac{\left(9 * \sqrt{10}\right) + \left(12 * \sqrt{4}\right) - \left(6 * \sqrt{25}\right) - \left(8 * \sqrt{10}\right)}{\left(9 * \sqrt{25}\right) + \left(12 * \sqrt{10}\right) - \left(12 * \sqrt{10}\right) - \left(16 * \sqrt{4}\right)} \\ &= \frac{\sqrt{10} + (12 * 2) - (6 * 5)}{(9 * 5) - (16 * 2)} \\ &= \frac{\sqrt{10} - 6}{13}\end{aligned}$$

In addition, we have rationalized our denominator and written our answer in simplified form.

Linear Equations and Inequalities
Absolute Value Equations and Inequalities

1 Solving Linear Equations

For example, let us solve the following equation for *y*.

$$3*(y+2)=5*(y-6)$$

The first thing we will do is, we will distribute the 3 to both of the two terms inside the parenthesis, as well as the 5 to the two terms inside the parenthesis. Now, when solving linear equations, what we want to do now is bring all the variables to one side, all the numbers to the other and solve.

$$\begin{aligned} 3*(y+2)&=5*(y-6)\\ 3y+6&=5y-30\\ 6+30&=5y-3y\\ 2y&=36\\ \underline{y}&\underline{=18} \end{aligned}$$

All right, let us look at another example. Let us solve this equation for *t*.

$$4-3*(t+2)+t=5*(t-1)-7t$$

Again, we will start in a similar way, and distribute this -3 to both of these two terms, as well as the 5 to both of these two terms. Then combine like terms on both sides of the equation. Again, bring all the variables to one side, and the numbers to the other.

$$\begin{aligned} 4-3*(t+2)+t&=5*(t-1)-7t\\ 4\text{-}3t\text{-}6+t&=5t\text{-}5\text{-}7t\\ \text{-}2\text{-}2t&=\text{-}2t\text{-}5\\ \text{-}2t+2t&=\text{-}5+2\\ 0&=\text{-}3\ \ ??? \end{aligned}$$

What does that mean? That means that no value of *t* will work. No matter what value of *t* we try to plug into this equation, we will never get a true statement. What would our answer be in such a situation? Our answer would be 'No solution,' because there is no value of t that will satisfy this equation.

All right, let us see another example.

$$\frac{v+10}{15}-\frac{1}{5}=\frac{v+1}{6}-\frac{1}{10}$$

Now, this example is different in that we have fractions. There are different approaches to solving such an equation. What the approach that students seem to like best is to find the least common multiple of the denominators, therefore, we can multiply both sides of the equation by that least common multiple to eliminate the fractions. Let us do that. What is the least common multiple of the denominators? Is that not 30? Therefore, let us multiply both sides of the equation by 30.

$$\begin{aligned} \frac{v+10}{15}-\frac{1}{5}&=\frac{v+1}{6}-\frac{1}{10}\\ 30*\left(\frac{v+10}{15}-\frac{1}{5}\right)&=30*\left(\frac{v+1}{6}-\frac{1}{10}\right)\\ 30*\left(\frac{v+10}{15}\right)+30*\left(-\frac{1}{5}\right)&=30*\left(\frac{v+1}{6}\right)+30*\left(-\frac{1}{10}\right)\\ 2*(v+10)+6*(-1)&=5*(v+1)+3*(-1)\\ 2v+20-6&=5v+5-3\\ 2v+14&=5v+2\\ 5v-2v&=14-2\\ 3v&=12\\ \underline{v}&\underline{=4} \end{aligned}$$

2 Algebraic Symbol Manipulation

For example, let us solve the following equation for *F*.

$$C = \frac{5}{9} * (F - 32)$$

Right now we have C = to something, and we want to manipulate this equation to get F = to something instead. We will begin by multiplying both sides of the equation by $^9/_5$ in order to get rid of this fraction. Then we will add 32 to both sides of the equation.

$$\begin{aligned} C &= \frac{5}{9} * (F - 32) \\ \frac{5}{9}C &= F - 32 \\ F &= \frac{5}{9}C + 32 \end{aligned}$$

By the way, these are the equations that we use to convert between Fahrenheit and Celsius.

All right, let us see another example. Let us solve the following equation for *y*.

$$x = \frac{3y + 2}{y - 3}$$

What we can do is we can start by multiplying both sides of the equation by this denominator, y - 3. We will cancel, and we are assuming here, of course, that $y \neq 3$. Now let us distribute the x to both of these two terms. Remember that we want to solve this equation for *y*. Therefore, let us bring the entire *y* to one side, and everything else to the other. Now we will factor *y* out of both terms on the left, and we are assuming here, of course, that $x \neq 3$.

$$\begin{aligned} x &= \frac{3y + 2}{y - 3} \\ x * (y - 3) &= 3y + 2 \\ xy - 3x &= 3y + 2 \\ xy - 3y &= 2 + 3x \\ y * (x - 3) &= 2 + 3x \\ y &= \frac{2 + 3x}{x - 3} \end{aligned}$$

3 Linear Word Problems

A total of 703 tickets were sold for the school play. They were either adult or student tickets. There were 53 more student tickets sold than adult tickets. How many adult tickets were sold?

$$\begin{aligned} s &= \#\text{ of student tickets sold} \\ a &= \#\text{ of adult tickets sold} \end{aligned}$$

There are 53 more student tickets sold. Moreover, the total number of tickets sold was 703, and since they are either adult tickets or student tickets, that means that

$$\begin{aligned} s &= a + 53 \\ s + a &= 703 \end{aligned}$$

Now, we are asked to find out how many adult tickets were sold, which means we want to solve for *a*. Therefore, if we replace in this equation *s* with a + 53, then we will have an equation that just involves *a*, and we will be able to solve for it. Therefore, let us do that.

$$\begin{aligned} s &= a + 53 \\ s + a &= 703 \\ (a + 53) + a &= 703 \\ 2a + 53 &= 703 \\ 2a &= 650 \\ a &= 325 \end{aligned}$$

All right, let us look at another example.

James will rent a car for the weekend. He can choose one of two plans. The first plan has an initial fee of \$ 69, and costs an additional \$ 0.60 for every driven mile. Whereas, the second plan has no initial fee, but costs \$ 0.90 for every driven mile. How many miles would James need to drive for the two plans to cost the same?

Let us start by introducing some notations here.

$$\begin{aligned} x &= \#\text{ of miles James drives} \\ \text{cost of the first plan} &= 69 + .6x \\ \text{cost of the second plan} &= .9x \end{aligned}$$

We want to know how many miles he would need to drive for the two plans to cost the same. Therefore, we want to find the value of x that will make these equal. That is, we will solve the following equation for x.

$$\begin{aligned} 69 + .6x &= .9x \\ 69 &= .9x - .6x \\ 69 &= .3x \\ x &= \frac{69}{.3} \\ \underline{x} &\underline{= 230} \end{aligned}$$

4 Percentage Word Problems

Starting today, a table is being sold at Jaime's furniture store for \$ 345. This is 69 % of its regular price. What was the price yesterday?

Well, if today is the first day that this table is being sold at this discounted price, then the price yesterday was the regular price.

$$\mathrm{x} = \text{price yesterday}$$

Therefore, \$ 345 is 69 % of x, because the sale price is 69 % of the regular price. Translating this into an equation gives us:

$$\begin{aligned} 345 &= .69x \\ x &= \frac{345}{.69} \\ \underline{x} &\underline{= 500} \end{aligned}$$

All right, let us see another example.

At a sale this week, a desk is being sold for \$ 603. This is a 33 % discount from its original price. What is its original price?

Well, what does this mean, a 33 % discount? A 33 % discount means that the desk is being sold for 100 %- 33 %, or 67 % of its original price. Let x = its original price. This desk is being sold for \$ 603, which means \$ 603 is 67 % of its original price, or x. Again, let us translate this into an equation.

$$\begin{aligned} 603 &= .67x \\ x &= \frac{603}{.67} \\ \underline{x} &\underline{= 900} \end{aligned}$$

Therefore, the original price then of the desk was \$ 900.

5 Area and Perimeter Word Problems

For example, the length of a rectangle is five times its width. If the area of the rectangle is 405 square feet, find its perimeter. Okay, let l = length of the rectangle, and w = the width.

Now we are told that the length is five times its width. This means then, that l, the length, is five times the width of $5w$. However, we are also told that the area of the rectangle is 405 square feet.

$$\boxed{\begin{aligned} l &= 5w \\ A &= lw \\ lw &= 405 \\ (5w)w &= 405 \end{aligned}}$$

$$\begin{aligned} 5w^2 &= 405 \\ w^2 &= 81 \\ w &= \pm\sqrt{81} \\ &\underline{\underline{w = 9}}[ft] \end{aligned}$$

$$\begin{aligned} l &= 5 * 9 \\ &\underline{\underline{l = 45}}[ft] \end{aligned}$$

$$\begin{aligned} P &= 2l + 2w \\ P &= 2 * 45 + 2 * 9 \\ &\underline{\underline{P = 108}}[ft] \end{aligned}$$

All right, let us look at one more.

The length of a rectangle is 4 cm longer than its width. If the perimeter of the rectangle is 48 cm, find its area.

$$\begin{aligned} l &= w + 4 \\ P &= 2l + 2w \\ 2l + 2w &= 48 \\ 2(w + 4) + 2w &= 48 \\ 2w + 8 + 2w &= 48 \\ 4w &= 40 \\ &\underline{\underline{w = 10}}[cm] \end{aligned}$$

$$\begin{aligned} l &= w + 4 \\ l &= 10 + 4 \\ &\underline{\underline{l = 14}}[cm] \end{aligned}$$

$$\begin{aligned} A &= lw \\ A &= 10 * 14 \\ &\underline{\underline{A = 140}}[cm^2] \end{aligned}$$

6 Distance, Rate and Time

For example, two trains leave the station at the same time; one is heading west, and the other east. The westbound train travels 12 $^{mi}/_{h}$ faster than the eastbound train. If the two trains are 400 mi apart after 2 h, what is the rate of the westbound train?

$$\begin{aligned} x &= \text{rate, in miles per hour, of the weatbound train} \\ x - 12 &= \text{rate, in miles per hour, of the eastbound train} \\ \text{distance} &= \text{rate} * \text{time}\ (d = r * t) \end{aligned}$$

Now, we are going to use the famous formula, the distance = rate * time to help us solve this problem. Let us keep track of things in a table here.

	r	t	d
west	x	2	2x
east	x-12	2	2*(x-12)

However, we are told that after those 2 h, these two trains are 400 mi apart. This means, this total distance is 400. That is the sum of those two distances has to be 400. Namely,

$$\begin{aligned} 2x + 2(x - 12) &= 400 \\ 2x + 2x - 24 &= 400 \\ 4x &= 424 \\ \underline{x} &\underline{= 106}[mi] \end{aligned}$$

7 Mixture Problems using Linear Equations

For example, a student is mixing two solutions that contain HCl. The first solution has 15 % HCl, and the second has 5 % HCl. How many ml of each solution should the student mix in order to obtain 100 ml of an 8 % HCl solution?

$$\begin{aligned} x &= \#\ \text{ml of the 15\% solution} \\ y &= \#\ \text{ml of the 5\% solution} \\ x + y &= \#\ \text{ml of the 8\% solution} \end{aligned}$$

However, looking back up here, there is no problem. We know how many ml it has to be; it has to be 100, which means:

$$\begin{aligned} x + y &= 100 \quad or \\ y &= 100 - x \end{aligned}$$

Important to realize here is that the amount of HCL before mixing is equal to the amount of HCL after mixing. We do not create more HCL by mixing. Therefore, let us solve this linear equation for *x*.

$$\begin{aligned} .15x + .05 * (100 - x) &= .08 * 100 \\ .15x + 5 - .05x &= 8 \\ .1x &= 3 \\ \underline{x = 30}\ and \quad & \underline{y = 70} \end{aligned}$$

Therefore, writing our answer up here, the student would need to mix 30 ml of the 15 % solution with 70 ml of the 5 % solution in order to obtain 100 ml of the 8 % HCL-solution.

8 Solving a Linear Inequality

For example, let us solve the following inequality for *v*, and write our answer in interval notation.

For the most part, when we solve linear inequalities, we proceed in the same way as we do when we solve linear equalities. However, there is one difference, which we will see.

$$\begin{aligned} -7 + 4v &\geq -15 \\ 4v &\geq -8 \\ \underline{v} &\underline{\geq -2} \end{aligned}$$

When we multiply or divide both sides of an inequality by a positive number, the inequality stays the same, like in this case. When we multiply or divide both sides of an inequality by a negative number, we flip the inequality sign.

Here is -2; *v* can be = -2, and anything to the right. Now we are asked to put our answer in interval notation. Doing this, we have closed bracket -2, because we want to include -2, up to + infinity.

$$[-2, \infty)$$

Let us see another example. Let us solve the following inequality for *x*, and write our answer in interval notation.

$$\begin{aligned} -3x - 15 &> 3 \\ -3x &> 18 \\ \underline{x} &\underline{< -6} \end{aligned}$$

When we multiply or divide both sides of an inequality by a +number, the inequality stays the same, like in this case. When we multiply or divide both sides of an inequality by a -number, we flip the inequality sign.

Let us look at this again on the number line.

It is -6. We do not want *x* = -6, therefore, we put an open circle, and then less than everything to the left of -6. Again, we are asked to put our answer in interval notation.

$$(-\infty, -6)$$

Now, if we did not want to worry about dividing by this -, we could have started just a bit different here. We still have the same inequality.

$$\begin{aligned} -3x - 15 &> 3 \\ -15 - 3 &> 3x \\ -18 &> 3x \\ -6 &> x \\ \underline{x} &\underline{< -6} \end{aligned}$$

Therefore, we get -6 > *x*, which we can write right in the other direction, *x* < -6, which is our same answer.

9 Compound Inequalities

A compound inequality is two inequalities joined by either the word *and* or the word *or*. For example, this would be considered a compound inequality.

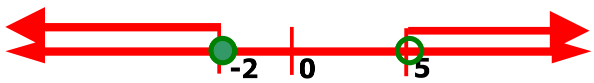
$$3x + 2 \le -4 \;\; or \;\; 4x + 4 > 24$$

The way we solve compound inequalities is we work with each side separately. Therefore, let us start with this first inequality here.

$$\begin{aligned} 3x + 2 &\le -4 \\ 3x &\le -6 \\ \underline{x} &\underline{\le -2} \end{aligned}$$

Now we solve the second inequality.

$$\begin{aligned} 4x + 4 &> 24 \\ 4x &> 20 \\ \underline{x} &\underline{> 5} \end{aligned}$$



We put a closed circle at -2, and less than we go to the left. *x* > 5, therefore, *x* cannot be = 5; we put an open circle, and then greater than we go to the right, which would be our answer.

All right, let us see another example.

$$15 \le 3 - 2x < 33$$

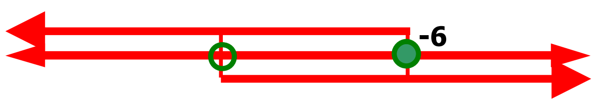
Now, although we do not see the word *and*, or the word *or* here, this is still a compound inequality. In fact, it is a short way of writing the following.

$$15 \le 3 - 2x \;\; and \;\; 3 - 2x < 33$$

Again, we solve both sides separately:

$$\begin{array}{ll} 15 \le 3 - 2x & \quad 3 - 2x < 33 \\ 12 \le -2x & \quad -2x < 30 \\ \underline{x \le -6} & \quad \underline{x > -15} \end{array}$$

Now let us graph this.



Therefore, $x \le -6$, we put a closed circle here, and go to the left, and *x* > -15, we put an open circle here, and go to the right. Now, these inequalities are joined by the word *AND*, our solution is all *x*-values that make both of them true. This would be the intersection here.

$$\underline{-15 > x \le -6}$$

10 Linear Inequality Word Problems

Rachel is a salesperson. She receives a salary of \$ 45,000 per year. In addition, she receives 7 % of her sales amount for the year. What are the sales amounts that will allow her to earn more than \$ 56,900 per year?

Therefore, we want Rachel's earnings to be > \$ 56,900. Well, what are these earnings? We are told that she receives a base salary of \$ 45,000, but then also 7 % for her sales amount for the year. That is her earnings, or the base salary, plus 7 %, or 0.07 of means times her sales amount.

$$\begin{aligned} \text{Rachels's earnings} &> 56{,}900 \\ \textit{Base} &+ .07\,(\text{sales amount}) \\ x &= \text{Rachel's annual sales amount} \\ 45{,}000 + .07x &> 56{,}900 \\ .07x &> 11{,}900 \\ \underline{x} &\underline{> 170{,}000} \end{aligned}$$

Rachel's annual sales amount is larger than \$ 170,000; indeed, she will be earning more than \$ 56,900 per year.

11 Absolute Value Equations

For example, let us solve this equation for *w*.

$$|2w + 5| = 7$$
$$2w + 5 = 7 \quad \textit{or} \quad 2w + 5 = -7$$

In other words, if |A| = B, then either A = B or A = -B which is what we are using here. Let us work with each equation separately.

$$|2w + 5| = 7$$
$$\begin{aligned} 2w + 5 &= 7 & \textit{or} \qquad 2w + 5 &= -7 \\ 2w &= 7\text{-}5 & 2w &= \text{-}7\text{-}5 \\ 2w &= 2 & 2w &= \text{-}12 \\ \underline{w} &\underline{= 1} & \underline{w} &\underline{= \text{-}6} \end{aligned}$$

Now let us check these answers by plugging them into the original equation.

$$\begin{aligned} &w = 1{:} && |2*(1) + 5| = 7 && |7| = 7 \\ &w = -6 > && |2*(-6) + 5| = 7 && |-7| = 7 \end{aligned}$$

When solving these types of absolute value equations, and the right-hand side is a positive umber like this, and we apply this fact, then we do not really need to check our answers, but it never hurts.

However, with something like this, where the right-hand side is not a positive number, we must check our answers. We are still going to use the fact that, if |A| = B, then either A = B or A = -B, but we are just going to have to check our answers at the end.

Okay, applying this:

$$|x + 4| = 3x + 8$$
$$\begin{aligned} x + 4 &= 3x + 8 & \textit{or} \qquad x + 4 &= -3x - 8 \\ -2x &= 4 & 4x &= -12 \\ \underline{x} &\underline{= -2} & \underline{x} &\underline{= -3} \end{aligned}$$

Now remember, we must check these answers by plugging each of them into the original equation wherever we see an *x*.

$$\begin{aligned} &x = -2{:} && |(-2) + 4| = 3*(-2) + 8 && |2| = 2 \\ &\cancel{x = -3{:}} && |(-3) + 4| = 3*(-3) + 8 && |1| = -1 \end{aligned}$$

As you can see, the first equation is true, but the second is *not*. This means we need to cross off -3. Therefore, our only answer is x = -2.

Just be very careful to check your answers when that right-hand side is not a positive number!

12 Absolute Value Inequalities

Let us solve the following inequality for *x* and put our answer in interval notation.

$$|5-2x| \le 9$$

If the absolute value is ≤ 9, that means, what is inside the absolute value is between -9 and 9. In other words, we are going to use the following factor to help us.

$$\begin{gathered} |y| \le a \\ -a \le y \le a \end{gathered}$$

Let us apply that here.

$$\begin{gathered} -9 \le 5-2x \le 9 \\ -14 \le -2x \le 4 \\ 7 \ge x \ge -2 \\ \underline{-2 \le x \le 7} \end{gathered}$$

We are asked to put our answer in interval notation. Therefore, our answer would be closed bracket.

$$[-2,7]$$

All right, let us look at another example. Let us solve this inequality, and again put our answer in interval notation.

$$\begin{gathered} |3+2x| > 1 \\ 3+2x > 1 \quad or \quad 3+2x < -1 \\ 2x > 1-3 \qquad 2x < -1-3 \\ \underline{x > -1} \qquad \underline{x < -2} \end{gathered}$$

Done! Now, the difference here is that we have greater than rather than less than like the last example. If the absolute value is > 1, then what is inside the absolute value is either > 1 or < -1. In other words, we are going to use the following fact to help us solve this problem.

$$\begin{gathered} y > a \\ y > a \quad or \quad y < -a \end{gathered}$$

Again, we are asked to put our answer in interval notation, and doing this gives us our answer of:

$$(-\infty,-2) \cup (-1,\infty)$$

13 Inequalities Involving Radicals

For example, let us solve the following inequality for *x*.

$$\sqrt{(x-6)^2} \le 9$$

All right, what is the left-hand side here equal? Most students will say that it *x* - 6, which is not true. In fact, it is = |*x*| - 6.

That is, the square root of something squared is only equal to that thing, if that thing is ≥ 0, otherwise, it is the - of it. Alternatively, the $\sqrt{y^2}$ is equal to $|y|$.

Therefore, be very careful here, it is a very common mistake. For example:

$$\begin{gathered} \sqrt{y^2} = \begin{cases} y, y \ge 0 \\ -y, y < 0 \end{cases} \\ \sqrt{y^2} = |y| \\ \sqrt{2^2} = \sqrt{4} = 2 \\ \sqrt{(-2)^2} = \sqrt{4} = 2 \end{gathered}$$

This translates into this compound inequality.

$$\sqrt{(x-6)^2} \le 9$$
$$|x\text{-}6| \le 9$$
$$-9 \le x-6 \le 9$$
$$\underline{-3 \le x \le 15}$$
$$\underline{\underline{[-3,15]}}$$

Just be really careful that you use the absolute value here.

Let us see another example. Let us solve this inequality for *y*.

$$\sqrt{(3-2y)^2} > 5$$

Again, we have the square root of something squared. Be careful that the number is the absolute value of that thing, which needs to be > 5. When solving absolute value inequalities, if we have the absolute value of something as > 5, then that means that the quantity is either > 5, or the quantity < -5.

Now we will solve this compound inequality by working each side separately.

$$\sqrt{(3-2y)^2} > 5$$
$$|3-2y| > 5$$
$$3-2y/5 \qquad \textit{or} \qquad 3-2y|\text{-}5$$
$$\text{-}2y/2 \qquad\qquad\qquad \text{-}2y|\text{-}8$$
$$\underline{y < \text{-}1} \qquad\qquad\qquad \underline{y > 4}$$
$$\underline{\underline{(-\infty,-1)\cup(4,\infty)}}$$

Just be very careful that you use this absolute value when starting.

Quadratic Equations, Rational Equations, Additional Equation Solving Techniques

1 Factoring a Quadratic Expression

For example, let us factor

$$x^2 - 10x + 21$$

To factor this expression means we want to write it as a product, ($x+m$) * ($x+n$), where m and n are integers. Now, think about what must be true. Let us 'FOIL' this right-hand side

$$x^2 + nx + mx + mn$$
$$x^2 + (n+m) * x + mn$$

In order for this equality to hold, the coefficient of x on the left, namely -10, has to be equal to the coefficient of x on the right, n + m, and then these constants.

$$n + m = -10$$
$$mn = 21$$

Therefore, let us start by looking at factor of 21, and make a table here.

m	n	n + m
1	21	22
-1	-21	-22
3	7	10
-3	-7	-10

What are some factors of 21? We have 1 and 21, we have -1 and -21, we have 3, and 7, and -3, and -7. When we add these values together, we will see, that the first three rows do not fit, because they do not give -10. Therefore, we find our m and n up here. Our answer then is x -3 * x -7.

This method of factoring is sometimes referred to as factoring by trial and error, because we are trying different factors of 21, until we find the ones that add up to -10.

Let us verify to ourselves that indeed we did get the right factorization here. If we 'FOIL' this, we get:

$$(x-3)*(x-7)$$
$$= x^2 - 7x - 3x + 21$$
$$\underline{= x^2 - 10x + 21}$$

This is sometimes called the ac-method, because we want to look at a quadratic expression of the form:

$$ax^2 + bx + c$$

Now, this ac-method is more commonly used when our $a > 1$.

For example, let us factor:

$$2x^2 - 7x + 3$$

Let us compare it to the quadratic form. We see here, that $a = 2$, $b = -7$, and $c = 3$. Let us apply the ac-method that was just described. What we do is we multiply a and c together, therefore, it is 2 * 3, which is 6. We want to look at factors of 6, that will add together to give us our b or -7, and we find that the factors would be -1 and -6. Then we factor by grouping. From the first two terms, we can factor out an x, and we are left with:

$$2x^2 - 7x + 3 = 2x^2 - x - 6x + 3$$
$$= x * (2x\text{-}1) \text{-} 3 * (2x\text{-}1)$$
$$\underline{= (2x\text{-}1) * (x\text{-}3)}$$

1.1 Special Factoring Formulas

For example, let us factor this expression. Now this is what we call the difference of two squares. In other words, this is actually:

$$64u^2 - 25 = (8u)^2 - (5)^2$$

So it is the difference of 2 squares. In this type of situation, there is a special formula that can help us factor this, and the formula is:

$$A^2 - B^2 = (A - B) * (A + B)$$

Okay, let us apply that here with A = 8u and B = 5.

$$\begin{aligned} 64u^2 - 25 &= (8u)^2 - (5)^2 \\ &= (8u - 5) * (8u + 5) \end{aligned}$$

Alright, let us see another special formula.

Let us doctor this expression.

$$4x^2 - 20x + 25$$

Now this is what we call a perfect square, there are special formulas here as well, and the formulas are:

$$\begin{aligned} A^2 + 2AB + B^2 &= (A + B)^2 \\ A^2 - 2AB + B^2 &= (A - B)^2 \end{aligned}$$

Now, a tip that our expression might be of this form is that both the first term and the last term are perfect squares.

$$\begin{aligned} 4x^2 - 20x + 25 &= (2x)^2 \text{-} 20x + (5)^2 \\ &= \underline{(2x - 5)^2} \end{aligned}$$

Sometimes it is hard for students to recognize these perfect square forms. If you did not notice, that this was of this form, you could have factored in other ways, and you would have gotten to the same answer, but it is very useful if you can recognize these forms.

Let us see another one. Let us factor this expression.

$$8x^3 + y^6$$

Now this is what we call a sum of two cubes. In other words, this is:

$$8x^3 + y^6 = (2x)^3 + (y^2)^3$$

Again, there is a special formula in this case, and the formula is:

$$\begin{aligned} A^3 + B^3 &= (A + B) * (A^2 - AB + B^2) \\ A^3 - B^3 &= (A - B) * (A^2 + AB + B^2) \end{aligned}$$

Notice on this first formula, we have a + on the left side, and a + on the first term of the right side, but a - on the second term. On the second expression we have it vice versa.

$$\begin{aligned} 8x^3 + y^6 &= (2x)^3 + (y^2)^3 \\ &= (2x + y^2) * \left((2x)^2 - (2x)(y^2) + (y^2)^2\right) \\ &= \underline{(2x + y^2) * (4x^2 - 2xy^2 + y^4)} \end{aligned}$$

It is very useful to be able to recognize these special factoring formulas. They can help you out a lot.

1.2 Factoring by Grouping

For example, let us factor the following expression by grouping.

$$4v^5 + v^4 + 20v + 5$$

To factor means, we want to write this expression as a product of other numbers or algebraic expressions. Now looking here, there is nothing in common to all four of these terms, and when that is the case, we try to group terms together, and look for common factors. We can begin by trying to group the first two terms together, and the last two terms together.

$$4v^5 + v^4 + 20v + 5 = \left(4v^5 + v^4\right) + \left(20v + 5\right)$$

The greatest common factor in the first two terms is v^4. Factoring that out, we are left with (4v + 1). The greatest common factor in the second two terms is 5, factoring that out, we are left with (4v + 1), which is the same binomial expression as in the first grouping. This is why factoring by grouping works here, because now we can factor that out of both of these.

$$\begin{aligned} 4v^5 + v^4 + 20v + 5 &= \left(4v^5 + v^4\right) + \left(20v + 5\right) \\ &= \underline{\left(4v + 1\right) * \left(v^4 + 5\right)} \end{aligned}$$

Let us look at another example.

Again, let us factor this expression by grouping, and we will start in the same way. We will group the first two terms together as well as the last two terms.

$$5w^3 - 4w^2 - 25w + 20 = \left(5w^3 - 4w^2\right) + \left(-25w + 20\right)$$

Now, the greatest common factor in the first two terms is w^2, and when we factor that out, we are left with ($5w$ - 4). What is the greatest common factor in this second grouping? Well, we get factor either a 5 out or a -5 out.

$$\begin{gathered} -25w + 20 = \\ 5 * \left(-5w + 4\right) \textit{ or } \quad -5 * \left(5w - 4\right) \end{gathered}$$

However, remember, our hope is that we are going to get the same binomial leftover. If we factor out a -5, we are left with ($5w$ - 4), which is that same binomial. That is what we want to factor out, the -5, therefore, this is -5 * ($5w$ - 4). Now we can factor this binomial out of each of these groupings, which gives us:

$$\begin{aligned} 5w^3 - 4w^2 - 25w + 20 &= \left(5w^3 - 4w^2\right) + \left(-25w + 20\right) \\ &= w^2 * \left(5w - 4\right) - 5 * \left(5w - 4\right) \\ &= \underline{5\left(w - 4\right) * \left(w^2 - 5\right)} \end{aligned}$$

2 Solving a Quadratic Equation

For example, let us solve this equation here for x.

$$x^2 + 4x - 5 = 0$$

Now, in solving quadratic equations it is useful to have 0 on one side, and see if we can factor the other. Here we already have 0 on the right, therefore, let us see if we can factor the left-hand side, which we can.

$$\begin{gathered} x^2 + 4x - 5 = 0 \\ \left(x + 5\right) * \left(x\text{-}1\right) = 0 \end{gathered}$$

Now we can use the fact that, if we have a product of factors = 0, then either the first factor is = 0, or the second factor is = 0, or both.

$$\begin{gathered} A * B = 0 \\ A = 0 \quad \textit{or} \qquad B = 0 \end{gathered}$$

Let us use that here. We have a product of factors = 0, which means, either the first factor is $x + 5 = 0$, or the second factor is $x - 1 = 0$.

$$\begin{gathered} x^2 + 4x - 5 = 0 \\ \left(x + 5\right) * \left(x\text{-}1\right) = 0 \\ x + 5 = 0 \qquad \textit{or} \qquad x\text{-}1 = 0 \\ \underline{x = \text{-}5 \textit{ or } \qquad x = 1} \end{gathered}$$

Alright, let us look at another example. Let us solve this equation for *y*.

$$2y^2 + 5y = 3$$

Now, this is a bit different from the first example in that we do not already have 0 on the right-hand side. Therefore, let us start off by bringing the 3 to the left-hand side.

$$\begin{gathered} 2y^2 + 5y = 3 \\ 2y^2 + 5y - 3 = 0 \end{gathered}$$

Now we have 0 on the right-hand side. Therefore, let us see if we can factor the left-hand side here. Moreover, we can:

$$\begin{gathered} 2y^2 + 5y = 3 \\ 2y^2 + 5y - 3 = 0 \\ (2y-1) * (y+3) = 0 \\ 2y - 1 = 0 \quad or \quad y + 3 = 0 \\ \underline{y = \frac{1}{2} \quad or \quad y = -3} \end{gathered}$$

2.1 Discriminant

Let us find the number of real solutions of each of these three quadratic equations.

$$\begin{aligned} &1) \quad 3x^2 - 6x + 5 = 0 \\ &2) \quad 3x^2 - 6x + 1 = 0 \\ &3) \quad 3x^2 - 6x + 3 = 0 \end{aligned}$$

Let us first recall the quadratic formula.

Quadratic Formula

The solution to the quadratic equation $ax^2 + bx + c = 0$ are given by the following.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, here we are not asked to find the solutions, but rather the number of real solutions. In addition, this quantity here under this square root is what determines that. This quantity is what we call the discriminant.

Discriminant : $b^2 - 4ac$

$b^2 - 4ac > 0$ two real solutions
$b^2 - 4ac = 0$ one real solution
$b^2 - 4ac < 0$ no real solution

Therefore, let us compute this discriminant for each of our 3 equations here.

$$\begin{aligned} 1) \quad & 3x^2 - 6x + 5 = 0 \\ & a = 3,\ b = -6, c = 5 \\ & b^2 - 4ac = (-6)^2 - 4(3)(5) \\ & \qquad = 36 - 60 \\ & \qquad = -24 < 0 \\ & \qquad \underline{\text{no solution}} \end{aligned}$$

2) $3x^2 - 6x + 1 = 0$

$a = 3,\ b = -6, c = 1$

$b^2 - 4ac = (-6)^2 - 4(3)(1)$

$= 36 - 12$

$= 24 > 0$

2 real solution

3) $3x^2 - 6x + 3 = 0$

$a = 3,\ b = -6, c = 3$

$b^2 - 4ac = (-6)^2 - 4(3)(3)$

$= 36 - 36$

$= 0$

1 real solution

Therefore, the discriminant helps us determine how many real solutions we have to a given quadratic equation.

2.2 Quadratic Formula

For example, let us use the quadratic formula to solve this equation for *x*.

$$3x^2 + x - 2 = 0$$

Here is the quadratic formula.

Quadratic Formula

The solution to the quadratic equation $ax^2 + bx + c = 0$ are given by the following.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$3x^2 + x - 2 = 0$

$a = 3, b = 1, c = -2$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{-1 \pm \sqrt{1 + 24}}{6} = \frac{-1 \pm 5}{6}$$

$$x = \frac{-1 + 5}{6} \quad or \quad x = \frac{-1 - 5}{6}$$

$$x = \frac{2}{3} \quad or \quad x = -1$$

Let us see another one. Let us use the quadratic formula to solve for *y*.

$2y^2 + 5y - 1 = 0$

$a = 2, b = 5, c = -1$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-5 \pm \sqrt{33}}{4}$$

$$x = \frac{-5 + \sqrt{33}}{4} \quad or \quad x = \frac{-5 - \sqrt{33}}{4}$$

However, we could have solved this by completing the square, and we would have arrived at the same answers that the quadratic formula just gave us.

2.3 Solving a Quadratic Equation – Completing the Square

Let us solve this equation:

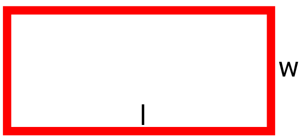
$$4x^2 + 8x + 1 = 0$$

The first thing we will do is bring the constant term, namely the 1, to the right-hand side of the equation. Now, when completing the square it is very important to make sure that the coefficient of the square term is = 1. Right now, it is = 4. Therefore, what we need to do is divide both sides of the equation by 4. The next step then in completing the square is, that we take ½ the coefficient of *x*, which in this case is 2, and ½ of 2 is 1. Then we square this number, therefore, we get $1^2 = 1$, and then we add this to both sides of the equation. Now the left-hand side is a perfect square.

$$\begin{aligned}
4x^2 + 8x + 1 &= 0 \\
4x^2 + 8x &= -1 \\
x^2 + 2x &= -\frac{1}{4} \\
(x+1)^2 &= \frac{3}{4} \\
x + 1 &= \pm\sqrt{\frac{3}{4}} \\
x &= -1 \pm \frac{\sqrt{3}}{2} \\
\underline{x = -1 + \frac{\sqrt{3}}{2} \quad or \quad x = -1 - \frac{\sqrt{3}}{2}}
\end{aligned}$$

3 Quadratic Equation Word Problems

For example, the perimeter of a rectangle is 22 feet, and the area is 24 feet^2. Let us find the dimensions of the rectangle.



We are given that the perimeter is 22 feet, and recall that the perimeter of a rectangle is:

$$\begin{aligned}
2l + 2w &= P \\
2l + 2w &= 22 \\
l * w &= A \\
l * w &= 24
\end{aligned}$$

Now we can divide both sides of this perimeter equation by 2. Then we can plug into this area equation, which would give us an equation that just involves this *w*, namely:

$$\begin{aligned}
l * w &= 24 \\
(11-w) * w &= 24 \\
11w - w^2 &= 24 \\
w^2 - 11w + 24 &= 0 \\
(w-3)*(w-8) &= 0 \\
\underline{w = 3 \quad or \quad w = 8}
\end{aligned}$$

Now we can plug this value into.

$$\begin{aligned}
l + w &= 11 \\
l &= 11 - w \\
l &= 11 - 3 \qquad or \qquad l = 11 - 8 \\
\underline{l = 8 \qquad or \qquad l = 3}
\end{aligned}$$

Therefore, in either case, a rectangle is 3 feet by 8 feet, which are the dimensions that we are looking for.

Alright, let us look at another example. A model rocket is launched with an initial upward velocity of 30 $^{m}/_{s}$. The rocket's height after *t* s is given by this equation here.

$$h = 30t - 5t^2$$

Find all values of *t* for which the rockets height is 10 m.

We are launching this rocket from the ground, and we want to know the time at which its height is 10 m. Then it is going to come back down, and we want to know the other time at which its height from the ground is 10 m. We can find these times by plugging 10 in for *h*, and solving for *t*.

$$\begin{aligned}
h &= 30t - 5t^2 \\
10 &= 30t - 5t^2 \\
\frac{5t^2}{5} - \frac{30t}{5} - \frac{10}{5} &= \frac{0}{5} \\
t^2 - 6t + 2 &= 0 \\
t &= \frac{6 \pm \sqrt{(-6)^2 - 4(1)(2)}}{2} \\
t &= \frac{6 \pm \sqrt{28}}{2} \\
t &= \frac{6 \pm \sqrt{4 * 7}}{2} \\
t &= \frac{6 \pm 2 * \sqrt{7}}{2} \\
\underline{t} &\underline{= 3 \pm \sqrt{7}}
\end{aligned}$$

If we put these values into our calculator, we get that the first expression gives approximately 0.35 s, and the second gives approximately 5.657 s. Therefore, the rockets height is 10 m at approximately $t = 0.35$ s on the way up, and then its height from the ground is 10 m again at approximately $t = 5.65$ s. These would be the two values of *t* that we are looking for.

4 Simplifying Fractions

For example, let us simplify this expression here.

$$\frac{\frac{3+x}{15x}}{\frac{x^2-9}{5}}$$

Now, these types of fractions, where the numerator or the denominator or both contain a fraction, are often referred to as either complex fractions or compound fractions. However, do not confuse the word complex here with the complex number. Remember, when we are dividing two fractions, we multiply the numerator by the reciprocal of the denominator. That is, we flip the bottom fraction and multiply. When we multiply fractions, we multiply the numerators, and we multiply the denominators.

$$\begin{aligned}
\frac{\frac{3+x}{15x}}{\frac{x^2-9}{5}} &= \frac{3+x}{15x} * \frac{5}{x^2-9} \\
&= \frac{5 * (3+x)}{15x * (x^2-9)} \\
&= \frac{5 * (3+x)}{15x * (x+3) * (x-3)} \\
&\underline{= \frac{1}{3x * (x-3)}}
\end{aligned}$$

Alright, let us see another example. Let us simplify this fraction.

$$\frac{\frac{x}{y} - 2 + \frac{y}{x}}{\frac{x}{y} - \frac{y}{x}}$$

Now, there are different approaches here. For example, we could work with the numerator separately, and then work with the denominator separately. However, the method that students seem to like best, when working with this type of fraction, is first to eliminate.

$$\frac{\frac{x}{y} - 2 + \frac{y}{x}}{\frac{x}{y} - \frac{y}{x}} = \frac{\frac{x}{y} - 2 + \frac{y}{x}}{\frac{x}{y} - \frac{y}{x}} * \frac{xy}{xy}$$

$$= \frac{\left(\frac{x}{y}\right) * (xy) + (-2) * (xy) + \left(\frac{y}{x}\right) * (xy)}{\left(\frac{x}{y}\right) * (xy) + \left(\frac{-y}{x}\right) * (xy)}$$

$$= \frac{x^2 - 2xy + y^2}{x^2 - y^2}$$

$$= \frac{(x-y)^2}{(x-y)*(x+y)}$$

$$= \frac{x-y}{x+y}$$

4.1 Simplifying a Ratio of Polynomials

For example, let us simplify this expression here.

$$\frac{3x^2 + 3x - 18}{x^2 + 4x + 3}$$

The first thing we can do is factor 3 out of the numerator. Now we can factor both quadratic expressions in the numerator and denominator, and then we can cancel these common factors of x + 3.

$$\frac{3x^2 + 3x - 18}{x^2 + 4x + 3} = \frac{3*(x^2 + x - 6)}{x^2 + 4x + 3}$$

$$= \frac{3*(x+3)*(x-2)}{(x+3)*(x+1)}$$

$$= \frac{3*(x-2)}{x+1}$$

Alright, let us look at another example. Let us simplify this expression here.

$$\frac{4 - 4w^2}{w^2 + w - 2}$$

Again, we can begin by factoring out the common factor in the numerator, which is 4 here. Now we can factor about the numerator and denominator. Although these factors are not exactly the same, they are the negative of each other, are they not? In addition, to see this, we can distribute the negative, or -1, through to both of these terms.

$$w - 1 = -(1 - w)$$
$$w - 1 = -1 + w$$
$$w - 1 = w - 1$$

Now we can cancel these common factors of $1 - w$, and we are assuming here, of course, $w \neq 1$, which leaves us with our answer:

$$\begin{aligned}\frac{4-4w^2}{w^2+w-2} &= \frac{4*\left(1-w^2\right)}{w^2+w-2}\\ &= \frac{4*\left(1-w\right)*\left(1+w\right)}{w-1}\\ &= -\frac{4*\left(1+w\right)}{w+2}\end{aligned}$$

4.2 Adding and Subtracting Rational Expressions

For example, let us add these two rational expressions together.

$$\frac{5}{x-1}+\frac{4}{x-2}$$

Just like when we add fractions we need to make sure, that the denominators are equal before we add the numerators. Moreover, here we see that they are different, which means, we need to find the LCD. In this case, it is the product!

$$LCD = \left(x-1\right)*\left(x-2\right)$$

Now we need to multiply both the numerator and denominator by what is missing from the LCD. All right, and by commutativity, $x - 1 * x - 2$ and $x - 2 * x - 1$ are equal. This means, we can add the numerators, and now we can distribute both 5 and 4, which gives us:

$$\begin{aligned}\frac{5}{x-1}+\frac{4}{x-2} &= \left(\frac{5}{x-1}*\frac{\left(x-2\right)}{\left(x-2\right)}\right)+\left(\frac{4}{x-2}*\frac{\left(x-1\right)}{\left(x-1\right)}\right)\\ &= \frac{5*x-2+4*x-1}{\left(x-1\right)*\left(x-2\right)}\\ &= \frac{5x-10+4x-4}{\left(x-1\right)*\left(x-2\right)}\\ &= \frac{9x-14}{\left(x-1\right)*\left(x-2\right)}\end{aligned}$$

Let us see another one.

$$\frac{y}{y^2+2y-3}-\frac{1}{y^2-3y+2}+\frac{5}{y^2+y-6}$$

Again, we notice that these denominators are all different, which means we need to find the LCD in order to combine the numerators. However, what we will need to do first is factor the denominators to see what that LCD is. Again, we are going to build up these rational expressions by multiplying both numerator and denominator by what is missing from the LCD. By commutativity, these denominators are all equal, which means we can combine these into one fraction now. Then we are distributing and combining like terms in the numerator.

$$\begin{aligned}\frac{y}{y^2+2y-3}-\frac{1}{y^2-3y+2}+\frac{5}{y^2+y-6} &= \frac{y}{\left(y+3\right)*\left(y-1\right)}-\frac{1}{\left(y-1\right)*\left(y-2\right)}+\frac{5}{\left(y+3\right)*\left(y-2\right)}\\ \rangle\ LCD &= \left(y+3\right)*\left(y-1\right)*\left(y-2\right)\\ &= \frac{y}{\left(y+3\right)*\left(y-1\right)}*\frac{\left(y-2\right)}{\left(y-2\right)}-\frac{1}{\left(y-1\right)*\left(y-2\right)}*\frac{\left(y+3\right)}{\left(y+3\right)}+\frac{5}{\left(y+3\right)*\left(y-2\right)}*\frac{\left(y-1\right)}{\left(y-1\right)}\\ &= \frac{y*\left(y-2\right)-1*\left(y+3\right)+5*\left(y-1\right)}{\left(y+3\right)*\left(y-1\right)*\left(y-2\right)}\\ &= \frac{y^2-2y-y-3+5y-5}{\left(y+3\right)*\left(y-1\right)*\left(y-2\right)}\\ &= \frac{y+4}{\left(y+3\right)*\left(y-1\right)}\end{aligned}$$

4.3 Multiplying Rational Expressions

For example, let us multiply these two rational expressions, and simplify our answer.

$$\frac{9b}{8y} * \frac{4y^3}{3by}$$

Now, when multiplying rational expressions, we proceed in the same way as we do when we multiply numerical fractions. That is, we multiply the numerators, then we multiply the denominators and simplify.

$$\begin{aligned}\frac{9b}{8y} * \frac{4y^3}{3by} &= \frac{(9b)*(4y^3)}{(8y)*(3by)} \\ &= \frac{36 * b * y^3}{24 * b * y^2} \\ &= \frac{3y}{2}\end{aligned}$$

Let us look at another example. Let us multiply these two rational expressions.

$$\begin{aligned}\frac{x+1}{x^2-x-6} * \frac{x^2-4x+3}{4x+4} &= \frac{(x+1)*(x^2-4x+3)}{(x^2-x-6)*(4x+4)} \\ &= \frac{(x+1)*(x-3)*(x-1)}{(x-3)*(x+2)*(x+1)} \\ &= \frac{x-1}{4(x+2)}\end{aligned}$$

4.4 Dividing Rational Expressions

For example, let us divide:

$$\frac{2x^4y^2}{5y} \div \frac{4x}{15y}$$

Now, dividing by this is the same as multiplying by its reciprocal. When we multiply fractions, we multiply the numerators, and then multiply the denominators.

$$\begin{aligned}\frac{2x^4y^2}{5y} \div \frac{4x}{15y} &= \frac{2x^4y^2}{5y} * \frac{15y}{4x} \\ &= \frac{30x^4y^3}{20xy} \\ &= \frac{3x^3y^2}{2}\end{aligned}$$

Let us look at another example. Let us divide these two rational expressions.

$$\frac{4x-4}{2x+3} \div \frac{2x^2-x-1}{2x^2+5x+3}$$

Again, we will begin by multiplying by the reciprocal of this. We multiply the numerators, multiply the denominators, and then we will factor.

$$\begin{aligned}\frac{4x-4}{2x+3} \div \frac{2x^2-x-1}{2x^2+5x+3} &= \frac{4x-4}{2x+3} * \frac{2x^2+5x+3}{2x^2-x-1} \\ &= \frac{4*(x-1)*(2x+3)*(x+1)}{(2x+3)*(2x+1)*(x-1)} \\ &= \frac{4(x+1)}{2x+1}\end{aligned}$$

5 Square Root Property

Let us solve this equation for *x*.

$$9x^2 - 16 = 0$$

Now there are a few different ways to solve this. First of all, let us notice that the left-hand side is the difference of two squares, is it not? Moreover, remember, in this case we have a special formula for factoring the difference of two squares.

$$A^2 - B^2 = (A - B) * (A + B)$$

Let us apply that here. Now we have a product of factors = 0, which means, either the first factor is zero, or the second factor is zero.

$$\begin{aligned} 9x^2 - 16 &= 0 \\ (3x)^2 - (4)^2 &= 0 \\ (3x - 4) * (3x + 4) &= 0 \\ \underline{x = \frac{4}{3} \quad \text{or} \quad x = \frac{-4}{3}} \end{aligned}$$

Now rather than solving this equation this way, I could use the following property.

> Square Root Property
>
> The solutions to the equation $C^2 = D$, where D is a positive real number, are given by the following.
>
> $C = \pm\sqrt{D}$

Now how could we use this here? Well, let us start by adding 16 to both sides of this equation, and then dividing both sides by 9. In addition, by this property we get that the solutions to this equation.

$$\begin{aligned} 9x^2 - 16 &= 0 \\ 9x^2 &= 16 \\ x^2 &= \frac{16}{9} \\ x &= \pm\sqrt{\frac{16}{9}} \\ \underline{x} &\underline{= \pm\frac{4}{3}} \end{aligned}$$

Let us look at another example. Solve this equation for *y*.

$$(2y - 5)^2 = 9$$

Now, here it is much more straightforward just to start off using our square root property. Therefore, the solutions to this equation can be found as follows.

$$\begin{aligned} (2y - 5)^2 &= 9 \\ 2y - 5 &= \pm\sqrt{9} \\ 2y - 5 &= \pm 3 \\ 2y - 5 = 3 \quad &or \quad 2y - 5 = -3 \\ 2y = 8 \quad &or \quad 2y = 2 \\ \underline{y = 4 \quad} &\underline{or \quad y = 1} \end{aligned}$$

5.1 Solving Rational Equations

For example, let us solve this equation for *x*.

$$3 = -\frac{6}{x-4}$$

The first thing we will do is multiply both sides of the equation by the denominator. The $x - 4$ on the right will cancel, as long as $x \neq 4$. Now we distribute the 3 to both of the two terms, then we will add 12 to both sides, and divide both sides by 3, gives us:

$$\begin{aligned} 3 &= -\frac{6}{x-4} \\ 3*(x-4) &= -6 \\ 3x - 12 &= -6 \\ 3x &= 6 \\ \underline{x} &\underline{= 2} \end{aligned}$$

Now, when solving rational equations, it is very important to check your answer and make sure, that it does not make the denominator zero, and it does not. The only excluded value is $x = 4$. Therefore, $x = 2$ would work and would be our answer.

Let us look at another example. Let us solve this equation for *y*.

$$1 - \frac{y-3}{y-2} = \frac{2y-3}{y-2}$$

Again, we will begin in a similar way and multiply both sides of the equation by the denominator ($y - 2$). Now we will distribute the $y - 2$, to both of the terms. Again we can cancel these common factors of ($y - 2$) on both sides, assuming of course that $y \neq 2$. Then we have - this whole quantity and distributing our - gives us:

$$\begin{aligned} 1 - \frac{y-3}{y-2} &= \frac{2y-3}{y-2} \\ (y-2)*\left(1 - \frac{y-3}{y-2}\right) &= (y-2)*\left(\frac{2y-3}{y-2}\right) \\ (y-2) + (y-2)*\left(-\frac{y-3}{y-2}\right) &= 2y-3 \\ (y-2)-(y-3) &= 2y-3 \\ y - y - 2 + 3 &= 2y - 3 \\ 2y &= 4 \\ \cancel{y = 2} \end{aligned}$$

However, remember that we have to check our answer and make sure, that it is not the excluded value. Sure enough it is. If $y = 2$, then we would be dividing by zero. Therefore, we need to cross off $y = 2$ as a possibility here, which means that there is no solution here.

All right, let us see one more example. Let us solve the following equations for *w*.

$$\frac{2w-1}{w^2+6w+9} = \frac{1}{w^2+3w} + \frac{2}{w}$$

Now what are we going to be multiplying both sides of this equation by? What we need to do is determine what the least common multiple of these denominators, or the least common denominator is, which we can do by factoring these denominators first. Doing this, we can factor out a *w*. Therefore, the least common denominator then, or the least common multiple of these denominators, is $w * (w + 3) * (w + 3)$. This is what we are going to need to multiply both sides of this equation by. Now we will distribute this entire product here to both of these terms. Again, we can cancel these common factors, both on the left and right, assuming of course that $w \neq -3$. As long as $w \neq 0$, we can also cancel these common factors. Then distributing the *w* and foiling these terms that the 2 w^2 will cancel.

Now we can bring all the *w*-terms to the left-hand side, and solve the equation.

$$\frac{2w-1}{w^2+6w+9}=\frac{1}{w^2+3w}+\frac{2}{w}$$
$$\frac{2w-1}{(w+3)*(w+3)}=\frac{1}{w*(w+3)}+\frac{2}{w} \qquad LCD = w*(w+3)*(w+3)$$
$$w*(w+3)*(w+3)*\left(\frac{2w-1}{(w+3)*(w+3)}\right)=w*(w+3)*(w+3)*\left(\frac{1}{w*(w+3)}+\frac{2}{w}\right)$$
$$w*(2w-1)=(w+3)+2*(w+3)*(w+3)$$
$$2w^2-w=w+3+2*(w^2+6w+9)$$
$$-w=w+3+12w+18$$
$$-14w=21$$
$$\underline{w=-\frac{3}{2}}$$

However, remember we have to check, and make sure that this does not make our denominators up there zero. Moreover, it does not, because the only excluded values would be $w = 0$ or $w = -3$. Therefore, this is our answer then.

For example, let us solve this equation for *y*.

$$8-\frac{7}{y+2}=-\frac{5}{y-1}$$

We will begin by multiplying both sides of the equation by the least common multiple of these denominators here, which would be their product. Now we will distribute this product to both terms. On the left-hand side, as long as $y \neq -2$, we can cancel all $y + 2$. On the right, as long as $y \neq 1$, we can cancel all $y - 1$. Now you 'FOIL' this and distribute the 8, the -7, and the -5 to their responding terms. Then combine like terms on the left, and bring everything to left-hand side. Factoring out the left-hand side gives us a product of factors = 0, which means either the first factor is zero, or the second factor is zero, which gives us:

$$8-\frac{7}{y+2}=-\frac{5}{y-1}$$
$$(y+2)*(y-1)*\left(8-\frac{7}{y+2}\right)=(y+2)*(y-1)*\left(-\frac{5}{y-1}\right)$$
$$(y+2)*(y-1)*8+(y-1)*-7=(y+2)*-5$$
$$(y^2+y-2)*8+(y-1)*-7=(y+2)*-5$$
$$8y^2+8y-16-7y+7=-5y-10$$
$$8y^2+y-9=-5y-10$$
$$8y^2+6y+1=0$$
$$(4y+1)*(2y+1)=0$$
$$\underline{y=-\frac{1}{4}} \qquad or \qquad \underline{y=-\frac{1}{2}}$$

When solving rational equations, it is very important at this point to check, and make sure that these values do not make these denominators zero, and they do not. The only excluded values would be -2 or 1, and neither of these values are -2 or 1. Therefore, both of these would work.

All right, let us look at another example. Let us solve this equation for x.

$$\frac{2}{(x-2)*(x-4)}=2+\frac{1}{x-4}$$

Again, we will look at what the least common multiple of these two denominators is, which would be their product. Let us multiply both sides of the equation by this product. Now distribute this product to both of these two terms, and as long as $x \neq 2$, we can cancel all $x - 2$ on the left. In addition, as long as $x \neq 4$, we can cancel all $x - 4$ on the left and on the right. Let us 'FOIL' this out. Now we will distribute the 2 through to all these terms, combing like terms on the right, and bringing all the terms to one side. In the left-hand side, we will factor out. Now we have a product of factors = 0, which means, either the first factor is zero, or the second factor is zero.

$$\frac{2}{(x-2)*(x-4)} = 2 + \frac{1}{x-4}$$
$$(x-2)*(x-4)*\frac{2}{(x-2)*(x-4)} = (x-2)*(x-4)*\left(2+\frac{1}{x-4}\right)$$
$$(x-2)*(x-4)*\frac{2}{(x-2)*(x-4)} = (x-2)*(x-4)*2+(x-2)*(x-4)*\left(\frac{1}{x-4}\right)$$
$$2 = (x-2)*(x-4)*2+(x-2)$$
$$2 = (x2-6x+8)*2+x-2$$
$$2 = 2x^2-12x+16+x-2$$
$$2 = 2x^2-11x+14$$
$$x^2-11x+12=0$$
$$(2x-3)*(x-4)=0$$
$$\underline{x=\frac{3}{2}} \quad or \quad \cancel{\underline{x=4}}$$

Remember, we have to check that these values do not make our denominator zero. Sure enough, when $x = 4$, both of these denominators will be zero. Four is an excluded value; we have to cross it off our list. Therefore, our only answer would be $x = {}^3/_2$.

5.2 Solving an Quadratic Equation in Form

For example, let us solve this equation for x.

$$x^4-13x^2+36=0$$

The first thing you will want to notice here is that x^4 is the square of x^2. Therefore, if we let $u = x^2$, we can substitute this into our equation. Our original equation here is quadratic in form, is it not? All right! How do we solve this equation? We can factor the left-hand side. Now we have a product of factors. A very common mistake that students make here is that they will think they are done at this point. However, we are not done, are we? This is what u equals, but we are asked to solve for x. We can do that by substituting back in these values of u into our equation, then taking the square root of each side, we get:

$$x^4-13x^2+36=0$$
$$(x^2)^2-13x^2+36=0$$
$$u^2-13u+36=0$$
$$(u-9)*(u-4)=0$$
$$u-9=0 \quad or \quad u-4=0$$
$$u=9 \quad or \quad u=4$$
$$x^2=9 \quad or \quad x^2=4$$
$$x=\pm\sqrt{9} \quad or \quad x=\pm\sqrt{4}$$
$$\underline{x=\pm 3} \quad or \quad \underline{x\pm 2}$$

Alright let us look at another example.

Let us solve this equation for y.

$$3y^{2/3}+2y^{1/3}-8=0$$

Again let $u = y^{1/3}$, which we then can substitute into our equation. Now we have a quadratic equation, which we can solve by factoring. We have a product of factors = 0, therefore, either the first factor is zero or the second factor is zero. Now adding 4 and dividing by 3. Again, be careful here. Do not think that you are done. Remember we have to solve for y. This is what u is equal to.

Therefore, we will take these values of u and substitute them back into the equation to solve for y, and now we can cube both sides.

$$3y^{2/3} + 2y^{1/3} - 8 = 0$$
$$3 * \left(y^{1/3}\right)^2 + 2y^{1/3} - 8 = 0$$
$$3u^2 + 2u - 8 = 0$$
$$(3u - 4) * (u + 2) = 0$$
$$3u - 4 = 0 \quad or \quad u + 2 = 0$$
$$u = \frac{4}{3} \quad or \quad u = -2$$
$$y^{1/3} = \frac{4}{3} \quad or \quad y^{1/3} = -2$$
$$\left(y^{1/3}\right)^3 = \left(\frac{4}{3}\right)^3 \quad or \quad \left(y^{1/3}\right)^3 = (-2)^3$$
$$\underline{y = \frac{64}{27}} \quad or \quad \underline{y = -8}$$

5.3 Solving Equations Involving Radicals

For example, let us solve this equation for *w*.

$$\sqrt{w + 5} = 4$$

Let us begin by squaring both sides of this equation, therefore, we can eliminate our radical. However, we have to be very careful when we square both sides of an equation that we check our answers at the end, because we could come up with extraneous solutions.

$$\sqrt{w + 5} = 4$$
$$\left(\sqrt{w + 5}\right)^2 = 4^2$$
$$w + 5 = 16$$
$$\underline{w = 11}$$

Remember, we have to check our answer. That is we need to plugging $w = 11$ into our original equation, to see if the equation is satisfied.

$$\sqrt{11 + 5} = 4$$
$$\sqrt{16} = 4$$

Therefore, $w = 11$ works and is our answer.

Alright, let us look at another example. Let us solve the following equation for *x*.

$$\sqrt{x + 31} = x + 1$$

Again, we will begin by squaring both sides of this equation, and then squaring out the right-hand side. Now we bring all the terms to one side. We know that we can factor the left-hand side, which gives us:

$$\sqrt{x + 31} = x + 1$$
$$\left(\sqrt{x + 31}\right)^2 = (x + 1)^2$$
$$x + 31 = x^2 + 2x + 1$$
$$x^2 + x - 30 = 0$$
$$(x - 5)(x + 6) = 0$$
$$x - 5 = 0 \quad or \quad x + 6 = 0$$
$$\underline{x = 5} \quad or \quad \underline{x = -6}$$

However, remember we have to check these answers.

$$\begin{array}{ll} x = 5 & x = -6 \\ \sqrt{5+31} = 5+1 & \sqrt{-6+31} = -6+1 \\ \underline{\sqrt{36} = 6} & \underline{\sqrt{25} \neq -5} \end{array}$$

Which means our only answer then is x = 5.

All right let us look at one more example. Let us solve this equation for *y*.

$$\sqrt{y+7} = 2 + \sqrt{3-y}$$

Again, we will begin by squaring both sides of the equation. Now, be careful squaring the entire right-hand side. Unfortunately, we are still left with a radical in our equation, therefore, we need to isolate that. We can do that by bringing all other terms to the left-hand side. Now let us divide both sides by 2, and square both sides of this equation. Now distribute our 4, bring all the terms to one side, and factor the left-hand side, therefore, we have a product of factors = 0, which means, either the first factor is zero, which means that *y* = 2, or the second factor is zero, which means *y* = -6.

$$\begin{aligned} \sqrt{y+7} &= 2 + \sqrt{3-y} \\ \left(\sqrt{y+7}\right)^2 &= \left(2 + \sqrt{3-y}\right)^2 \\ y+7 &= 4 + 4\sqrt{3-y} + (3-y) \\ 2y &= 4\sqrt{3-y} \\ y &= 2\sqrt{3-y} \\ y^2 &= \left(2\sqrt{3-y}\right)^2 \\ y^2 &= 4(3-y) \\ y^2 &= 12 - 4y \\ y^2 + 4y - 12 &= 0 \\ (y-2)(y+6) &= 0 \\ \underline{y = 2} \quad or \quad & \underline{y = -6} \end{aligned}$$

However, remember we have to check these answers.

$$\begin{array}{ll} y = 2 & y = -6 \\ \sqrt{2+7} = 2 + \sqrt{3-2} & \sqrt{-6+7} = 2 + \sqrt{3+6} \\ \sqrt{9} = 2 + \sqrt{1} & \sqrt{1} = 2 + \sqrt{9} \\ \underline{3 = 3} & \underline{1 \neq 5} \end{array}$$

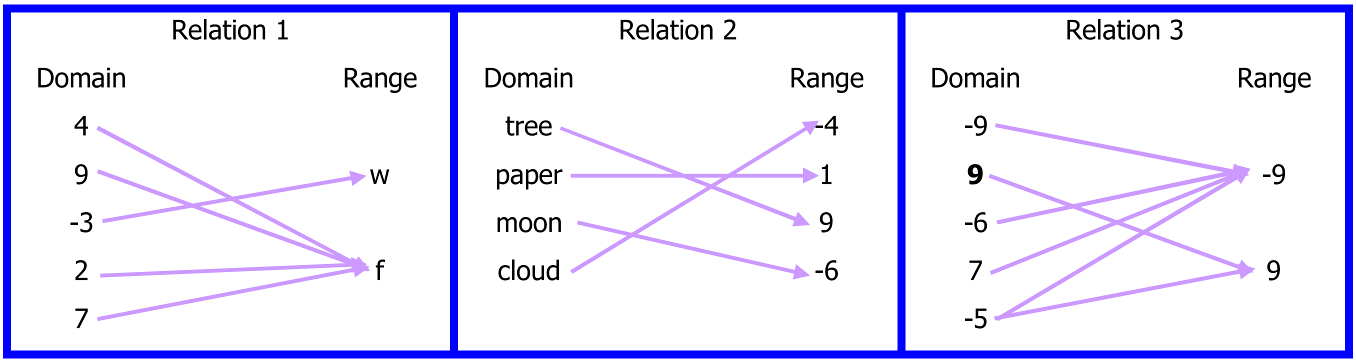
We need to cross off y = -6 here, which means our only solution then is *y* = 2.

Be careful when solving equations involving radicals, you have to always check your answers to make sure that you are not giving extraneous solutions.

Functions, Graphing Functions, Transformation of Functions

1 Relations versus Functions

For example, for each relation below, decide whether or not it is a function.



Now, a relation is a relationship between sets of information, but with a specific order. In other words, it is a set of ordered pairs. Each pair has a first component and a second component, and the set of first components is what we call the domain of the relation. The set of second components is what we call the range of the relations. Each value of the first component is sent to an element of the second component.

That is, we can write this first relation as the following set.

The value of 4 in the first component gets set to the value f in the second component, and 9 gets set to f, -3 gets set to w, 2 get set to f, and 7 gets set to f.

In the second relation tree gets set to 9, paper gets set to 1, moon gets set to -6, and cloud gets set to -4.

Finally, -9 gets set to -9, 9 gets set to 9, -6 gets set to -9, 7 gets set to -9, and -5 gets sent to -9, but also -5 gets sent to 9.

Now, a function is a special type of a relation. In a function, no two ordered pairs have the same first component. That is for each first component in a domain, there is exactly one second component. In order for a relation to be a function, for each first component in the domain, there is exactly one second component.

Now looking at our relations here, both relation 1 and relation 2 are functions, but relation 3 is not, because -5 is set to two different second components here, -9 and 9, as shown in the figure. Therefore, -5 is not to -9 as well as 9; this is not a function.

2 Functions and the Vertical Line Test

Let us determine whether the following equation defines y as a function of x.

$$2x - 3y = 6$$

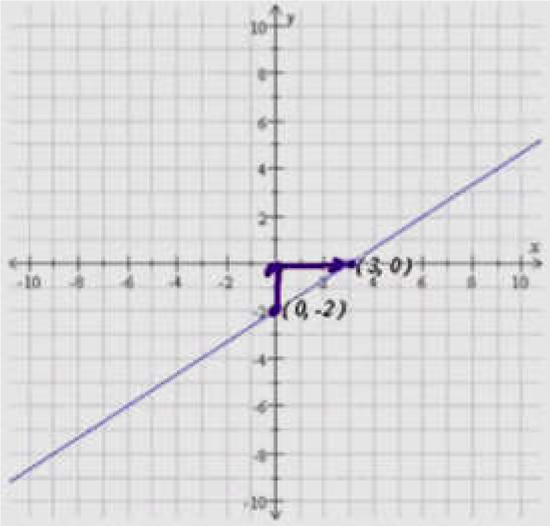
Now, remember that in an equation of two variables, if each value of the independent variable corresponds to exactly one value of the dependent variable, then the equation defines a function. In other words, if each allowed value of x maps to exactly one value of y, then the equation defines y as a function of x.

Solving our equations for y gives us:

$$2x - 3y = 6$$
$$3y = 2x\text{-}6$$
$$\underline{y = \frac{2}{3}x - 2}$$

Now, for each input of the independent variable x there will exactly one unique output y. Therefore, yes, this equation does define y as a function of x; y is a function of x.

Now, another way to reach this conclusion is to think of the graph. The graph of this equation is this line here. Is it not?



It has y-intercept at -2, and a slope of $^2/_3$, which means we go up 2 and over 3 neither, something called that vertical line test, which gives us a way to determine whether in a quotient defines *y* as function of *x* by looking at its graph.

What it said is that an equation defines a function, if each vertical line in the rectangular coordinate system passes through, at most, one point on the graph. Therefore, looking at our graph any vertical line that we draw will pass through.

This line will intersect the graph at only one point; it represents a function. The reason this is true, because what is the equation of a vertical line? It is *x* = something. Therefore, if it hits that graph at only one point that means for that *x*-value there is only one output *y*.

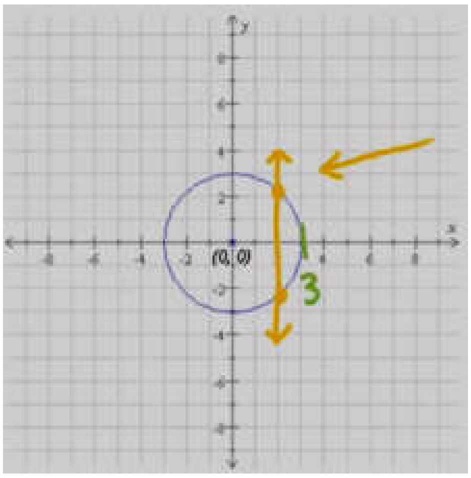
Let us look at another example. Let us determine whether this equation defines *y* as a function of *x*. Again, let us solve for *y*.

$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$

No, this equation does not define *y* as a function of *x*, because for whatever allowed value we put in here, we have two outputs, the + as well as the -. If we transform our equation, we will get:

$$y^2 = 9 - x^2$$
$$x^2 + y^2 = 9$$

Moreover, what does the graph of this look like?



It is just a circle centered at the origin with r = 3. If we pass a vertical line through this graph, it is going to intersect it in two places, is it not? Therefore, by the vertical line test, this equation does not define *y* as a function of *x*. This vertical line, that was just drawn here, is at *x* = 2. If we plug *x* = 2 in here, what do we get?

$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$
$$y = \pm\sqrt{9 - 2^2}$$
$$y = \pm\sqrt{9 - 4}$$
$$\underline{y = \pm\sqrt{5}}$$

This means that there are two different outputs. Therefore, it is not a function.

3 Domain of a Function

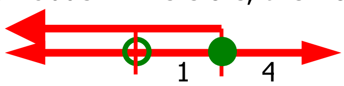
For example, let us find the domain of this function, and write our answer in interval notation.

$$f(x) = \frac{\sqrt{4 - x}}{x - 1}$$

Now, the domain of this function is the set of all *x* that make $\frac{\sqrt{4-x}}{x-1}$ a real number. The first thing to notice here is this x - 1 in the denominator. x - 1 cannot be zero. Otherwise, we would be dividing by zero, and this ratio would not be a real number. This means $x \neq 1$. Therefore, we will have to exclude $x = 1$ from the domain.

What else do we notice about this function? The numerator here contains a square root, and a square root will be a real number as long as what is under the square root, namely, $4 - x \geq 0$, or $x \leq 4$.

Let us take a look on the number line. $x = 1$ has to be excluded from our domain, therefore, let us put an open circle here at one. However, any other $x \leq 4$ will be in the domain. That is we will put a closed circle at 4 less than we go to the left, but we cannot include 1. Therefore, this would be our domain graphed here.



However, we are asked to put our answer in interval notation. Therefore, our answer would be:

$$(-\infty, 1) \cup (1, 4]$$

4 Evaluating Functions

For example, if *f* and *g* are defined as follows:

$$f(x) = 2 - 3x + x^2$$
$$g(x) = \frac{x + 4}{2x - 1}$$

Let us find all four of these:

$$f(-2)$$
$$f(x^2)$$
$$g(3)$$
$$g(\mathrm{x} + 1)$$

Now, we are given *f*(x). When we are evaluating this function at any input, we put the input here and here.

$$f(\text{input}) = 2 - 3(\text{input}) + (\text{input})^2$$

Let us write that out.

$$f(x) = 2 - 3x + x^2$$
$$f(-2) = 2 - 3(-2) + (-2)^2$$
$$f(-2) = 2 + 6 + 4$$
$$\underline{f(-2) = 12}$$

$$f(x) = 2 - 3x + x^2$$
$$f(x^2) = 2 - 3(x^2) + (x^2)^2$$
$$\underline{f(x^2) = 2\text{-}3x^2 + x^4}$$

Now, what about the function *g*? We are given *g*(x). When we are evaluating *g* at any input, we put the input into the nominator as well as into the denominator.

$$g(\text{input}) = \frac{(\text{input}) + 4}{2(\text{input}) - 1}$$

Therefore, let us write that out as well.

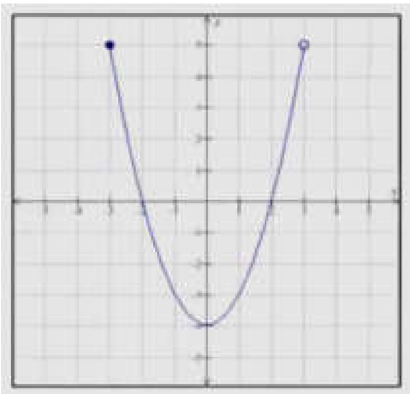
$$g(x) = \frac{x+4}{2x-1}$$
$$g(3) = \frac{3+4}{2(3)-1}$$
$$\underline{g(3) = \frac{7}{5}}$$

$$g(x+1) = \frac{x+1+4}{2(x+1)-1}$$
$$= \frac{x+5}{2x+2-1}$$
$$\underline{= \frac{x+5}{2x+1}}$$

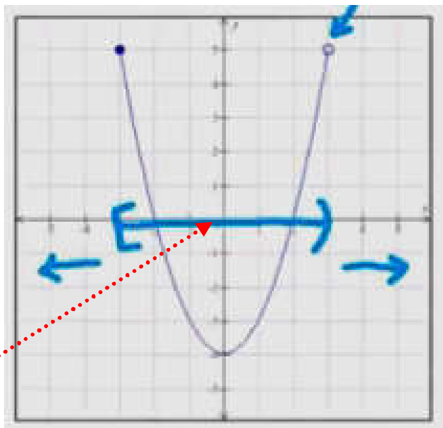
In addition, this is how we evaluate functions. Whatever we see inside the parenthesis, we put in as our input.

5 Finding the Domain and Range from a Graph

For example, here is the graph of *f*. Let us find its domain and range.

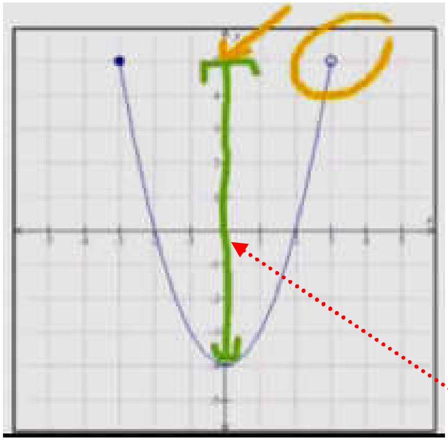


Now, remember that the graph of *f* is the set of all points (x, y) such that, y = f(x). The domain is the set of all x-coordinates of points in the graph, and the range is the set of all y-coordinates and points on the graph. Looking at our graph, what is the set of all possible x-coordinates of points on the graph?



It is this interval here, is it not? Think of projecting the graph onto the x-axis, and we will get this interval. Notice that if we go further to the right, there is no point on the graph with these x-coordinates, or if we go further to the left, there is no point on the graph with these x-coordinates either. Therefore, our domain is this interval, -3 up to 3, and notice that this is an open parenthesis here, because this point over x = 3 is open. In other words, x = 3 is not the x-coordinate of any point on that graph.

Now what about the range? What is the set of all possible y-coordinates of points on the graph? Let us look at our graph again.

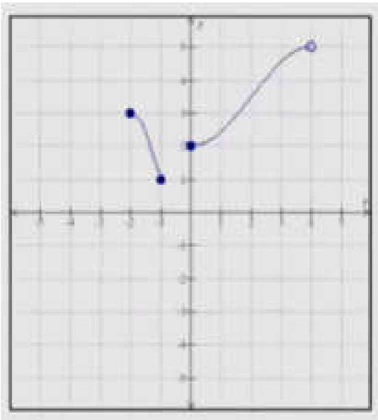


The set of all possible y-coordinates of points on the graph is this interval here. Think of projecting the graph onto the y-axis. Therefore, the range then is the interval -4 up to 5.

Now, some students think that we should not include 5 in our range. However, if you look over the y-axis there is a point on the graph with the y-coordinate = 5, which is why we include 5 here. Therefore, these would be our answers here.

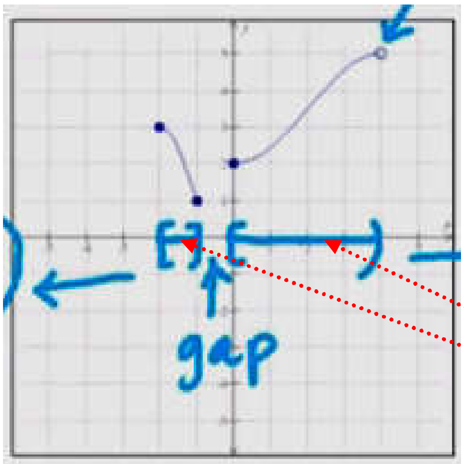
$$\text{Domain} = [-3,3)$$
$$\text{Range} = [-4,5]$$

All right, let us look at another example. Here is the graph of *g*. Let us find its domain and range.



Again the graph is the set of all ordered pairs *x*, *y* such that $y = g(x)$, and again the domain is this set of all possible x-coordinates of points on the graph, and the range is the set of all y-coordinates of points in the graph.

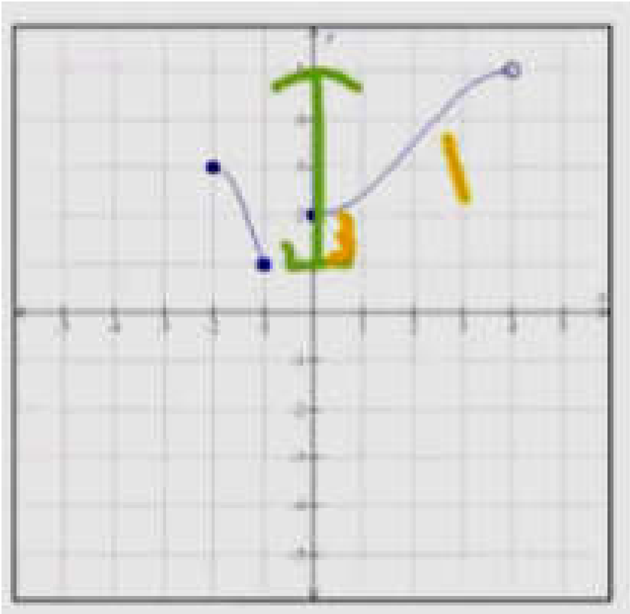
What is the domain? What is the set of all possible x-coordinates of points on our graph? Therefore, here is our graph.



Again, think of projecting the graph on to the x-axis. Therefore, the domain then would be these both intervals here. That is the domain equal to -2 to -1 union 0 to 4.

Notice a few things here. Again, $x = 4$ is not the x- coordinate of any point in our graph. Also, notice here there is this gap between -1 and 0, because those x-values are not the x-coordinate of any point on our graph. Neither are the x-values to the right of 4 or the x-values to the left of -2. Therefore, this is our domain.

Now what about the range? Again, looking at our graph, remember the range is the set of all possible y-coordinates of points on our graph.



Again, thinking of projecting this graph onto the y-axis will give us this interval. That is the range is equal to the interval from 1 to 5. We are not including 5 here, because 5 is not the y-coordinate of any point on our graph.

Now, sometimes students get confused, and they would not include this bottom portion, because they are only looking at this yellow marked piece of the graph. However, are these y-values here not y-coordinates for points over here? They are y-coordinates for points on our graph. Therefore, we include the entire interval from 1 up to 5.

Therefore, this is the domain and range of a function, given by its graph.

$$\text{Domain} = [-2,-1] \cup [0,4)$$
$$\text{Range} = [1,5)$$

6 Domain and Intercepts

For example, let us find the domain of f and any x- or y-intercepts of its graph.

$$f(x) = \sqrt{x+4} - 1$$

Let us start with the domain. What is under the square root, namely this $x + 4$ has to be ≥ 0, or subtracting 4 from both sides gives us that x has to be ≥ -4, which is the domain number. Let us write that in integral notation: Dom(f) = [-4,∞) which would be the domain.

What about the intercepts of its graph? Well, let us let $y = f(x)$. Let us start by determining whether the graph of f has any x-intercepts. Well, an x-intercept, if it exists, is the x-coordinate of a point where the graph of f intersects the x-axis. In addition, on the x-axis, y = 0, is it not? Therefore if we set $y = 0$ in our equation, we can see if there is any x-intercepts.

Therefore, the graph of f does have an x-intercept.

All right. What about the y-intercepts? A y-intercept, if it exists, is the y-coordinate of a point where the graph of f intersects the y-axis. Moreover, on the y-axis, what does x equal? It equals zero, does it not? We set $x = 0$ in our equation and solve for y. Therefore, the graph of f does have an y-intercept.

Now can the graph of a function have more than one x-intercept, or more than one y-intercept? There can be more than one x-intercept, but there cannot be more than one y-intercept, because, if this is indeed a function, then when $x = 0$, there cannot be two different outputs for y. Therefore, there can be at most 1 y-intercept.

Now a short summary:

$$y = f(x)$$
$$y = \sqrt{x+4} - 1$$

x – intercepts : $y = 0$

$$0 = \sqrt{x+4} - 1$$
$$1 = \sqrt{x+4}$$
$$1^2 = \left(\sqrt{x+4}\right)^2$$
$$1 = x + 4$$
$$\underline{x = -3}$$

y – intercepts: $x = 0$

$$y = \sqrt{0+4} - 1$$
$$y = 2 - 1$$
$$\underline{y = 1}$$

Alright, let us see another example. Again, let us find the domain of this function, *f*, and any x- or y-intercepts of it is graph.

$$f(x) = \frac{x^2 - 9}{x^2 - 4}$$

We will, begin with the domain again. The only issue with this function is, if the denominator is zero. When does $x^2 - 4 = 0$? This means that $x^2 = 4$ or $x = \pm 2$, therefore, we must exclude these values. That is the domain of *f*, again written in interval notation:

$$Dom(f) = (-\infty, -2) \cup (-2, 2)$$

Now we will find the intercepts:

$$y = f(x)$$
$$y = \frac{x^2 - 9}{x^2 - 4}$$

x – intercepts : $y = 0$

$$0 = \frac{x^2 - 9}{x^2 - 4}$$
$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$\underline{x = \pm 3}$$

y – intercepts : $x = 0$

$$y = \frac{0^2 - 9}{0^2 - 4}$$
$$\underline{y = \frac{9}{4}}$$

7 Difference Quotient

For example, let us compute the difference quotient for the function $\frac{f(x\text{-}h)\text{-}f(x)}{h}$, where $h \neq 0$.

This expression is called the difference quotient, and it is used often in calculus. Let us compute it for this function. $f(x) = 2x^2 - 4x + 3$

Still remember: $\boldsymbol{h \neq 0}$.

$$\begin{aligned} f(x) &= 2x^2 - 4x + 3 \\ f(x+h) &= 2(x+h)^2 - 4(x+h) + 3 \\ &= 2(x^2 + 2xh + h^2) - 4(x+h) + 3 \\ &= 2x^2 + 4xh + 2h^2 - 4x - 4h + 3 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(2x^2 + 4xh + 2h^2 - 4x - 4h + 3) - (2x^2 - 4x + 3)}{h} \\ &= \frac{2x2 + 4xh + 2h2 - 4x - 4h + 3 - 2x2 + 4x - 3}{h} \\ &= \frac{h(4x + 2h - 4)}{h} \\ &\underline{= 4x + 2h - 4} \end{aligned}$$

8 Piecewise-Defined Functions

A piecewise-defined function is a function that is defined in pieces or according to different rules depending upon the input.

For example, this would be considered a piecewise-defined function.

$$f(x) = \begin{cases} -3 \text{ if } x \neq -1 \\ -4 \text{ if } x = -1 \end{cases}$$

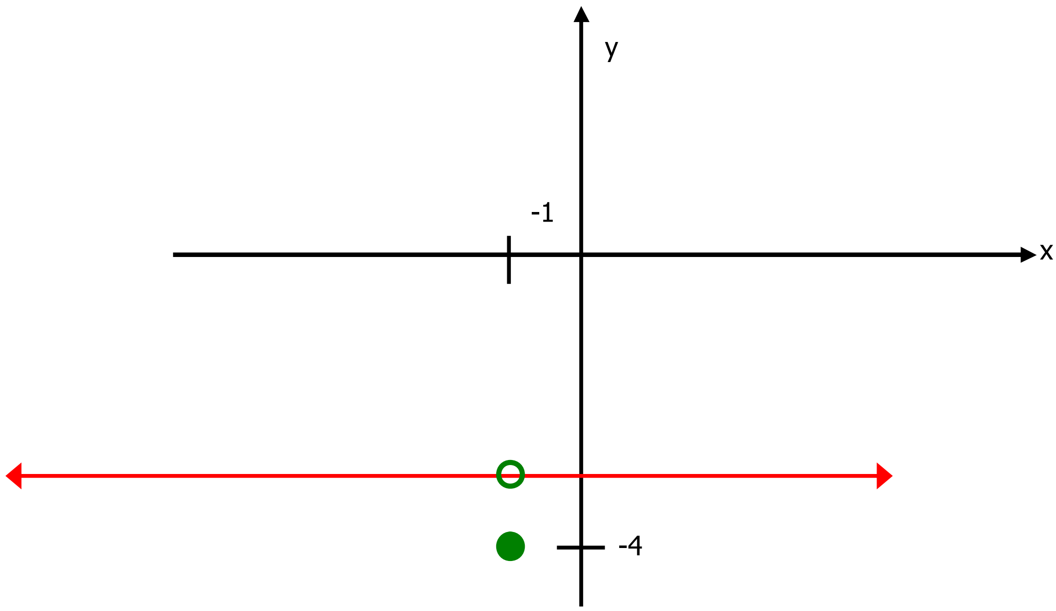
What does this mean? It means that *f* is defined by different rules according to what *x* is. That is, if $x \neq -1$, then **f**(*x*) =- 3. Otherwise, if *x* = -1, then **f**(*x*) = -4. Therefore, there are different rules that define *f* depending upon the input *x*.

Let us compute *f*(-5), *f*(-1), and *f*(2), and then we will graph *f*.

Since -5 ≠ -1, we will be using this first piece. What about *f*(-1)? -1 =- 1 causing the use of the second piece. Finally 2 ≠- 1 is causing the use of the first piece.

$$f(-5) = -3$$
$$f(-1) = -4$$
$$f(2) = -3$$

It remains for us now to graph *f*.



Therefore, this would be the graph of *f*.

Let us look at another example. Let *g* be defined by this piecewise function.

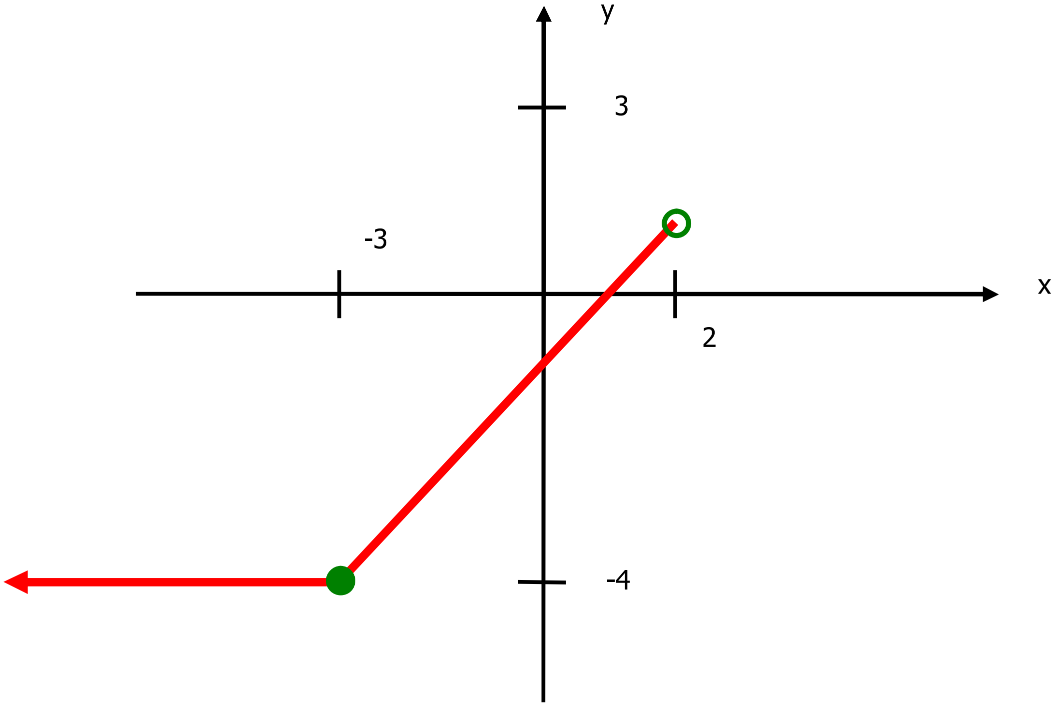
$$g(x) = \begin{cases} -4 & \text{if } x < -3 \\ x-1 & \text{if } -3 \leq x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$$

We are going to find *g*(-5), *g*(-2), and *g*(2), and then we will graph *g*.

Let us start with computing *g*(-5). Now, which of these three intervals does -5 lie in? It lies in the first interval, does it not? Because -5 < -3 we are going to use the first rule or the first piece to compute this. All right, what about *g*(-2)? Which piece or rule are we going to use to compute this? Well, -2 lies in the second interval, does it not? Therefore, we will use the second rule. Finally, what about *g*(2)? Which of the three intervals does 2 lie in? In addition, we have to be careful here, because we see two 2. However, the condition of equality is down in this third interval. Therefore, we use this third piece.

$$g(-5) = -4$$
$$g(-2) = -2 - 1 = -3$$
$$g(2) = 3$$

Okay. Therefore, it still remains to graph *g*.



9 Increasing / Decreasing / Constant

For example, the graph of the function *f* is shown below.

Let us find the intervals where *f* is increasing, decreasing, or constant, and then we will find the domain and range of *f* as well as any x- or y-intercepts of its graph.

Now we have the following:

- A function f is (strictly) **increasing** on an interval if, for all a and b in that interval, $a < b$ implies $f(a) < f(b)$. (See Figure 1 below.)
- A function f is (strictly) **decreasing** on an interval if, for all a and b in that interval, $a < b$ implies $f(a) > f(b)$. (See Figure 2 below.)
- A function f is **constant** on an interval if, for all a and b in that interval, $f(a) = f(b)$. (See Figure 3 below.)

A function *f* is said to be increasing or strictly increasing on an interval, if for all *a* and *b* in that interval $a < b$. The y-value at *a*, namely $f(a)$, is less than the y-value at *b*, namely $f(b)$.

What does that mean about the graph of the function? Looking down here in this figure, it means that the graph of the function is rising as we look from left to right.

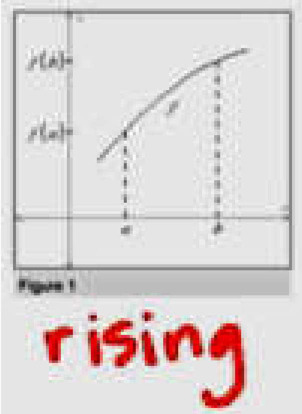


Figure 1

A function is said to be decreasing on an interval, if the y-value at *a* is larger than the y-value at *b*, for any value less than *b*. Moreover, looking down here, what does that mean about the graph? It means that the graph, looking from left to right, is falling.

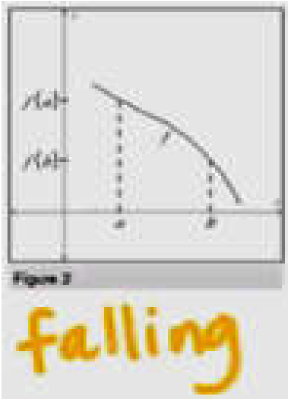
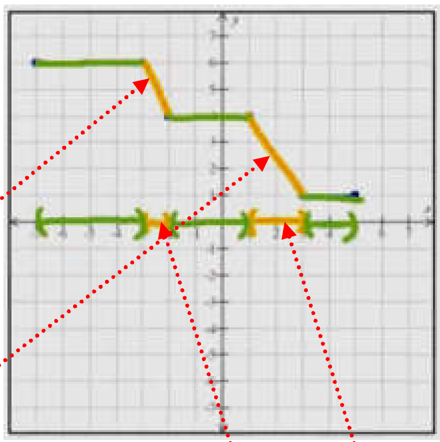


Figure 2

Now the function is said to be constant on an interval, if, for all *a* and *b* in that interval, the y-value $f(a)$ is equal to the y-value $f(b)$. Looking at the graph, the graph will be horizontal on that interval.

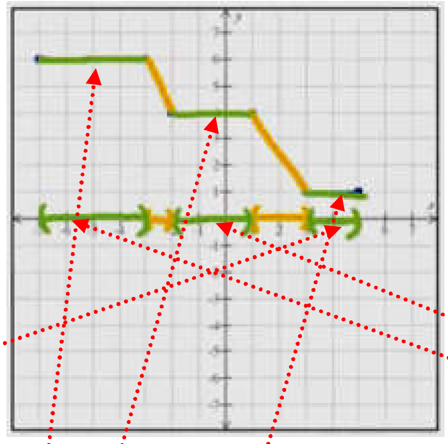
Figure 3

All right, let us start with increasing. Looking at the graph of our function here,



we see that the graph is not rising on any interval, therefore, our answer here would be none. Since the graph, is never rising. All right, where is *f* decreasing? Looking at our graph, we see that the graph is falling here, as well as here. This corresponds to these intervals here and here.

Now what about where this function is constant? Looking at our graph here,



we see that the graph is horizontal here, here, and here. This corresponds to the interval here, here, and here.

Summary:

$$\begin{array}{ll} \text{Increasing:} & \text{none (since the graph is never rising)} \\ \text{Decreasing:} & (-3,-2)\cup(1,3) \\ \text{Constant:} & (-7,-3)\cup(-2,1)\cup(3,5) \end{array}$$

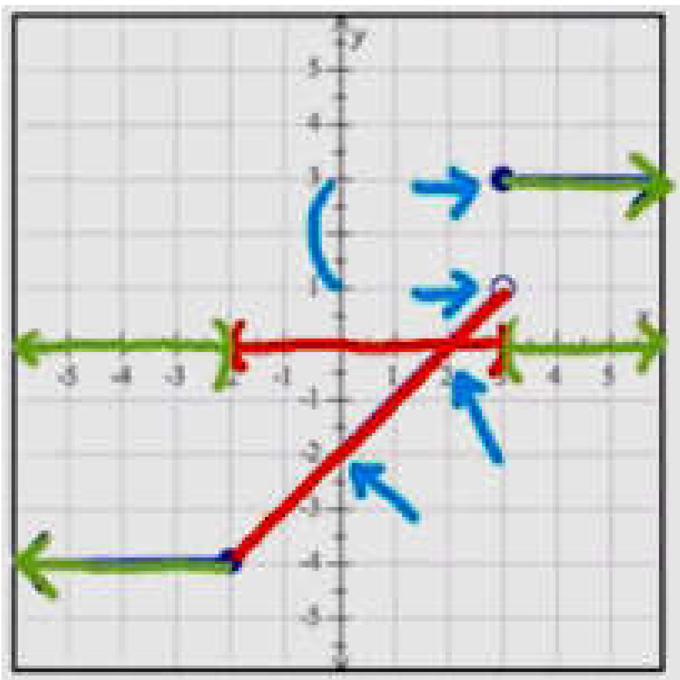
We are also asked to find the domain and range of *f* as well as any x- or y-intercepts of its graph. Looking at our graph, we see that the domain is equal to the interval from -7 up to 5, Remember that the domain is the set of all possible x-coordinates of points on the graph, and the range is a set of possible y-coordinates of points on the graph. Looking at our graph, the y-values are showing a range from 1 up to 6.

In addition, what is about x- and y-intercepts? This graph never touches the x-axis; therefore, there are no x-intercepts, and it crosses the y-axis at 4. The y-intercept is 4.

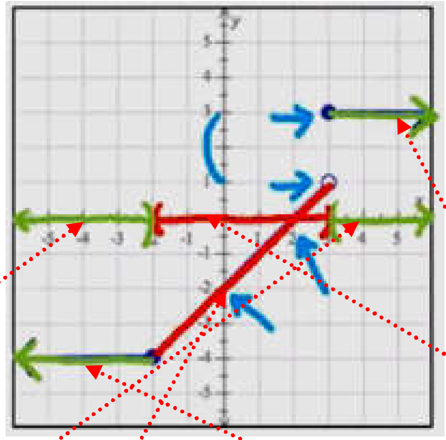
Summary:

$$\begin{array}{l} \text{Domain} = [-7,5] \\ \text{Range} = [1,6] \\ \text{y}-\text{int}: y = 4 \end{array}$$

All right, let us look at another example. Here is the graph of *f*, again we will determine the intervals where *f* is increasing, decreasing, or constant, and then we will find the domain and range, as well as any x- or y-intercepts.



Where is our graph increasing?



Looking at the graph, we see it is rising here. This corresponds to this interval here. Now what about decreasing? Looking at our graph, our graph is not falling on any interval, so this would be none. Finally, where is our function constant? Looking at our graph, we see it is horizontal here, as well as here, which corresponds to this interval as well as this interval. That is:

$$\begin{aligned} &\text{Increasing:} && (-2.3) \\ &\text{Decreasing:} && \text{none} \\ &\text{Constant:} && (-\infty,-2)\cup(3,\infty) \end{aligned}$$

Now notice that in both of these examples we have been written these intervals as open intervals. These terms, increasing, decreasing, and constant, are defined in terms of intervals. Not point-wise, but in terms of intervals, therefore, we could have included any of these endpoints, and our answer would be correct, but by convention most authors state these intervals as open intervals.

All right, it still remains to find the domain and range of *f* as well as any x- or y-intercepts of its graph. Therefore, the domain here is the set of all x-coordinates of points on the graph. Looking at our graph, every x-value has a corresponding point on the graph. That is the domain is the interval from -∞ to ∞.

What is ever with the range? We are missing a big piece there, are we not? There is no point on the graph that has a y-coordinate = 1, and there is a point on the graph that has y-coordinate = 3, and, in fact, every point to the right of this on the graph also has y-coordinates = 3. However, not only that, there are also no points on the graph that have y-coordinates > 3 or < -4.

All right, we see that we have both an x-intercept and an y-intercept.

Summary:

$$\begin{aligned} &\text{Domain} = [-\infty,\infty] \\ &\text{Range} = [-4,1)\cup[3,3] \\ &\text{x}-\text{int}: x = 2 \\ &\text{y}-\text{int}: y = -2 \end{aligned}$$

10 Graph Transformations

For example, let us sketch

$$y = \sqrt{x+2} + 1$$

using graph transformation.

Now, there are different types of graph transformations. There are transformations that we call translations, which are rigid movements of graphs, and then there are also transformations called stretching, shrinking, and reflecting, which are rigid movements of graphs.

Translations

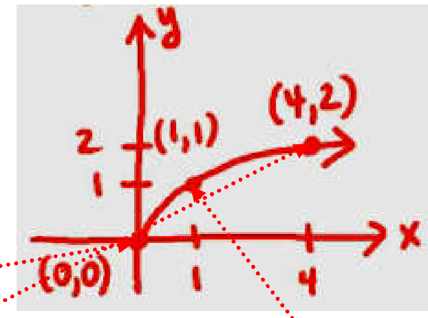
If f is a function and a and b are positive real numbers, then we have the following.

- To obtain the graph of $y = f(x-a)$, we shift the graph of $y = f(x)$ to the right a units.
- To obtain the graph of $y = f(x+a)$, we shift the graph of $y = f(x)$ to the left a units.
- To obtain the graph of $y = f(x)+b$, we shift the graph of $y = f(x)$ upward b units.
- To obtain the graph of $y = f(x)-b$, we shift the graph of $y = f(x)$ downward b units.

To sketch graphs using graph transformations, or these translations here, what we do is we start with what is called a base function. Looking here, we can start by sketching

$$y = \sqrt{x}$$

The graph looks like this.

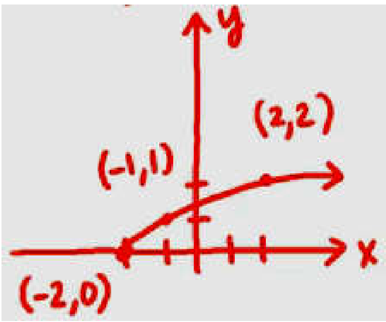


Here we have the origin (0, 0). What are some other points on this graph? Well, we know when $x = 1$, that y will also be = 1, which means, we know we have the point (1, 1) on our graph. Now, what other point lies on our graph? Well, we would not want to put, for example, $x = 2$ in here, because then the y-value would be √2. What is the next x-value that we know the square root of after 1 here? Well, that would be 4, would it not? Because we know that the √4 = 2. That is another point we have in our graph. Therefore, this is the point (4, 2).

All right, so let us sketch.

$$y = \sqrt{x+2}$$

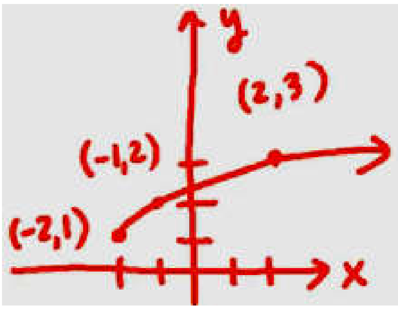
What is this +2 here do, under the square root, to this graph? Well, looking up there, we are in the second case, are we not? We are adding a +number to the x-value within the function, which means we are going to shift this graph to the left *a* units, or in this case two units. When we do that, what is going to happen to the point (0, 0)? The y-coordinate is not going to change, but the x-coordinate is going to have 2 subtracted from it, which means it is going to get translated to the point (-2, 0). What about the point (1, 1)? This is going to go to (-1, 1). In addition, what about the last point? This is going to go to (2, 2).



This shape is exactly the same, it is just moved two units to the left.

Now we have the following graph of $y = \sqrt{x+2} + 1$

Looking back up, what does this +1 do to the second graph here? Well, we are in the third case down there. We are now adding a +number to the y-value here on the second graph, which means we shift this second graph upward *b* units, or upward one unit. Therefore, when we do that, what happens to the point at (-2, 0)? This point goes to (-2, 1). What about the point (-1, 1)? This point is going to (--1 2.). Moreover, the last point (2, 2) is going to (2, 3).



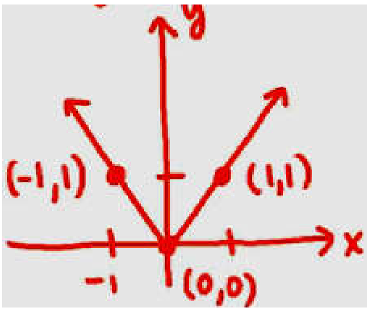
All right, let us look at another example. Let us sketch this graph here using graph transformations.

$$y = |x - 3| - 2$$

Again, we are going to use just translations here in this lesson. Looking at our function up here, what is our base function? It is

$$y = |x|$$

Therefore, let us sketch this.

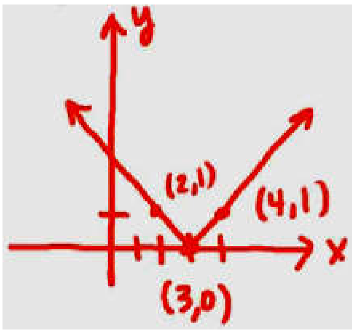


Looking back up to our equation, what does this -3 do to this graph here? Well, we are in the first case of translations, are we not? We are subtracting a +number from the x-coordinate inside the function, which means we can shift this graph to the right *a* units, or three units.

What is going to happen to the point (0, 0)? It is going to get moved to (3, 0). In this point, what is going to happen to (1, 1)? It is going to move to (4, 1). What about the point (-1, 1)? It is going to move to (2, 1).

That is, we have the following graph of: $y = |x + 3|$

This graph looks like this.

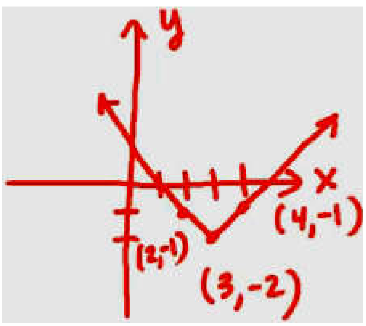


Finally, what does the -2 do to this second graph here? Well, we are down in this last case of translations. We are subtracting a +number from the y-value, which means we are going to shift this second graph downwards by *b* units, or by two units.

We shift this down two units. When we do that, what is going to happen to the point (3, 0)? This is going to go (3, -2). What about this point, (4, 1)? It is going to get shifted to (4, -1). What about the point (2, 1)? It is going to get shifted to (2, -1).

That is the graph we are looking for: $y = |x - 3| - 2$

Our graph looks like this.



Therefore our graph is being transformed by rigid translations in both these examples.

For another example, let us sketch $y = 3\text{-}2(x\text{-}1)^2$ by using graph transformations.

Now we already saw former on graph transformations, what the -1 and the 3 will do to the graph, but what does square and this 2 do? Well, we have the following.

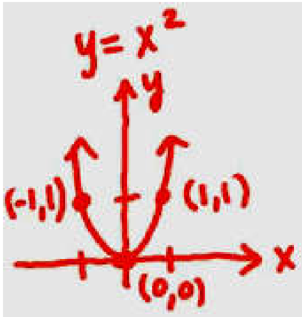
Stretching, Shrinking, and Reflecting

If f is a function and $c > 1$, then we have the following.

- To obtain the graph of $y = cf(x)$, vertically stretch the graph of $y = f(x)$ by a factor of c.
- To obtain the graph of $y = \frac{1}{c}f(x)$, vertically shrink the graph of $y = f(x)$ by a factor of c.
- To obtain the graph of $y = f(cx)$, horizontally shrink the graph of $y = f(x)$ by a factor of c.
- To obtain the graph of $y = f\left(\frac{1}{c}x\right)$, horizontally stretch the graph of $y = f(x)$ by a factor of c.
- To obtain the graph of $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- To obtain the graph of $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

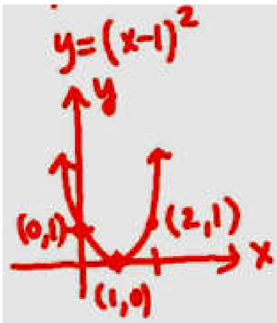
We still start with our base function;

$$y = x^2$$

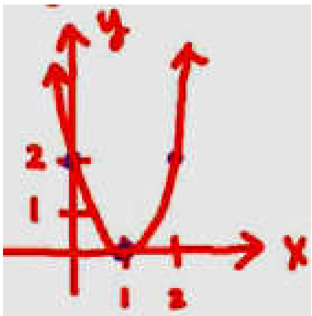


Remember, what does this -1 do inside the function? This rigidly shifts this graph one unit to the right. This means, to each x-coordinate we are adding 1, while leaving the y-coordinate alone. Which means that (0, 0) will move to (1, 0), (1, 1) will move to (2, 1), (-1. 1) will move to (0, 1).

Therefore $y = (x\text{-}1)^2$ will look like this.

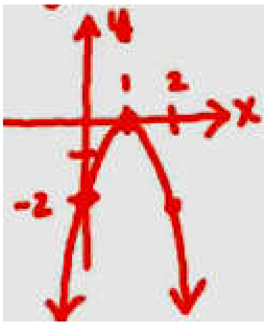


Now, let us discuss what the 2 in front of the parenthesis does to this graph. Well, we are in the first case up there. This means that we are going to vertically stretch this graph here by a factor of 2. That means what? This means for every y-coordinate that you see on the second graph, we are going to be multiplying that y-coordinate by 2. Therefore, the graph of $y = 2(x\text{-}1)^2$ will look like this.

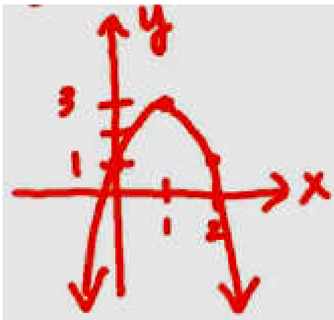


Looking back up to the function, now what does this - do there? Well, that means that we are going to reflect this graph about the x-axis.

That is $y = -2(x\text{-}1)^2$ will look like this.

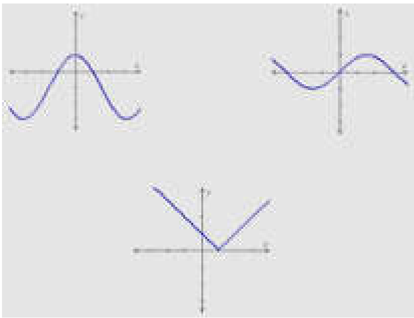


Finally, the graph we are looking for, this $y = 3\text{-}2(x\text{-}1)^2$ will be a vertical shift up three units of this graph. It is going to look like this.



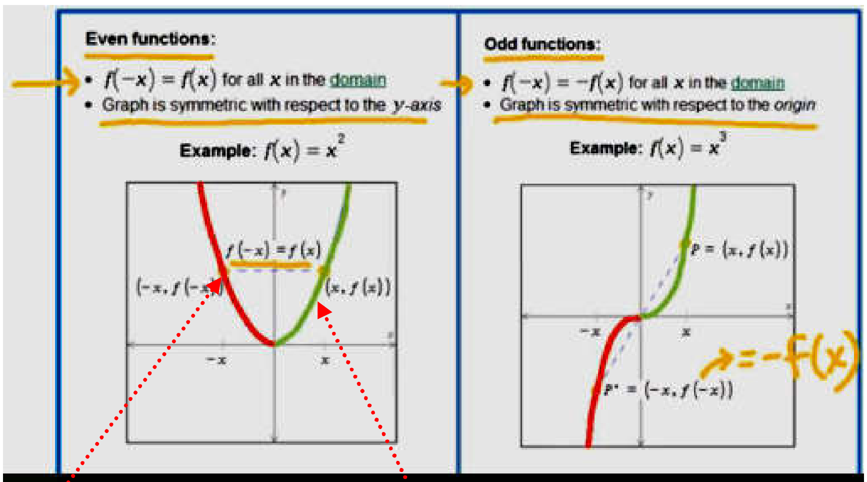
11 Odd and Even Functions

For example, the entire graph of three different functions is shown below.



Let us determine whether each function is odd, even, or neither.

A function is called:



For example, $f(x) = x^2$ is a famous example of an even function. Moreover, if you notice that, whenever x, $f(x)$ is on the graph, then $-x$, $f(-x)$ is also on the graph where $f(-x) = f(x)$. That is, plugging in opposite x-values yields the same y-value.

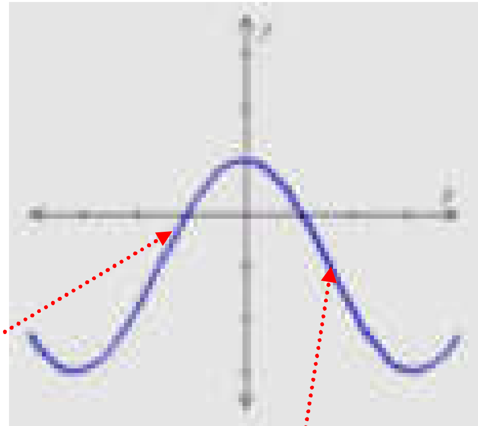
Therefore, this part of the graph is a mirror image of this part of the graph. That is, the graph is symmetric with respect to the y-axis.

A function is called odd, if $f(-x) = f(x)$ for all x in the domain. The graph of an odd function is symmetric with respect to the origin. For example, $f(x) = x^3$ is a famous example of an odd function. Notice, whenever x, $f(x)$ is on the graph, and so is $-x$, $f(-x)$. Moreover, $f(-x) = -f(x)$. In other words, when the x-values are opposites, the y-values are also.

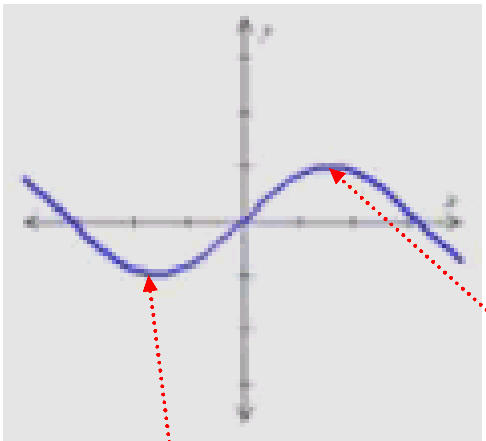
This means, that one part of the graph is a mirror image of a part of the other graph. In other words, its graph is symmetric with respect to the origin. Therefore, if we have the graph of the function, we can look at whether or not its graph is symmetric with respect to the y-axis or the origin or neither, to determine whether it is even, odd, or neither.

However, if we do not know the graph, and only are given the equation, then we can see that, plugging in opposite x-values, yields the same y-value, or opposite y-values, or neither.

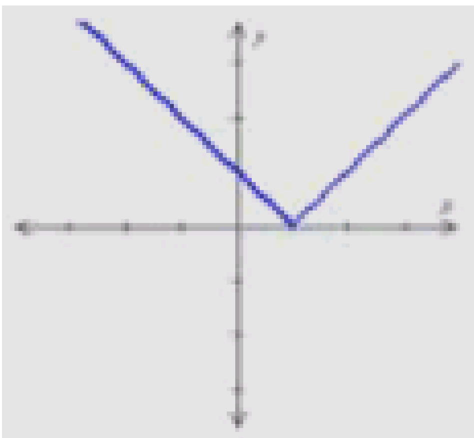
In this first example here, we are given the graph of three different functions. Let us look at this first function here.



We see that this part of the graph is a mirror image of this part of the graph. That is, its graph is symmetric with respect to the y-axis. Which means it is an even function.



Looking at our second function, we see that this part of the graph is a mirror image of this part of the graph. That is, it is symmetric with respect to the origin. Therefore, this is an odd function.

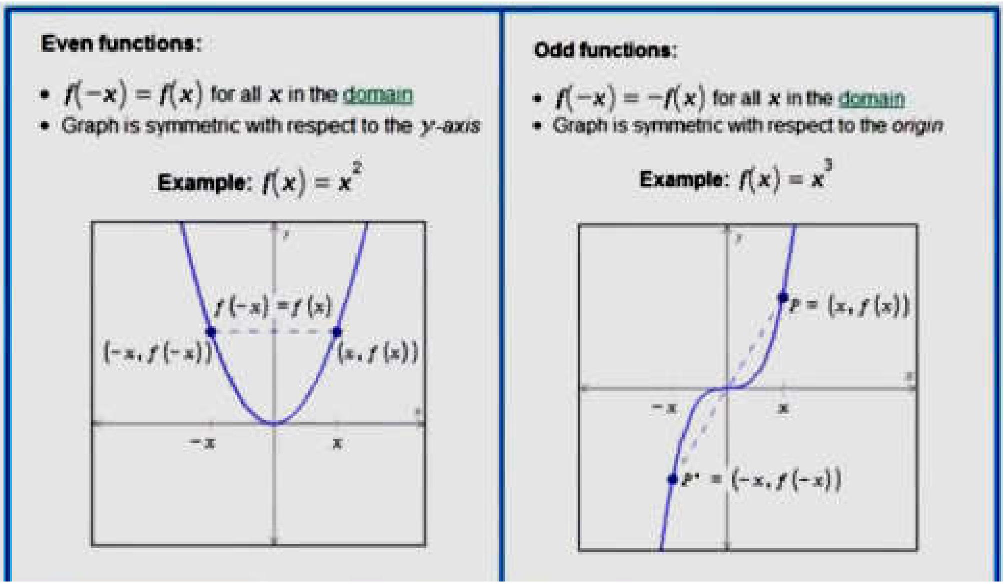


Finally, looking at our last graph here, it is not symmetric with respect to the y-axis or the origin, is it? Therefore, this would be neither odd nor even.

Let us look at another example. These functions are given:

$$g(x) = x^4 + 1$$
$$h(x) = x^5 + x^3$$
$$k(x) = x^6 + x^5$$

Let us determine whether each function is odd, even, or neither. Again, we have the following definitions.



However, now we are not given the graphs. We are only given the equations. Therefore, we want to see if plugging in opposite x-values yields the same y-value or plugging in opposite x-values yields opposite y-values, or neither of these.

$$\begin{aligned} g(x) &= x^4 + 1 \\ g(-x) &= (-x)^4 + 1 \\ &= x^4 + 1 \\ &= g(x) \qquad \text{EVEN} \end{aligned}$$

$$\begin{aligned} h(x) &= x^5 + x^3 \\ h(-x) &= (-x)^5 + (-x)^3 \\ &= -x^5 - x^3 \\ &= \left(x^5 + x^3\right) \\ &= -h(x) \qquad \text{ODD} \end{aligned}$$

$$\begin{aligned} k(x) &= x^6 + x^5 \\ k(-x) &= (-x)^6 + (-x)^5 \\ &= x^6 - x^5 \\ &\neq k(x) \\ &\neq -k(x) \qquad \text{NEITHER} \end{aligned}$$

Quadratic Functions, Inverse Functions, Polynomials

1 Equations of Quadratic Functions

For example, let us find the equation of the quadratic function whose graph passes through the point (-2, -1) and has a vertex at (-1, -3), and then we will put our answer in standard form.

We have the following vertex form for the equation of a quadratic function:

$$f(x) = a(x-h)^2 + K$$

Here we are given that the vertex is at (-1, --3), which means that h = -1 and K = -3, which we can plug into this vertex form:

$$\begin{aligned} f(x) &= a(x-h)^2 + K \\ f(x) &= a(x-(-1))^2 + (-3) \\ f(x) &= a(x+1)^2 - 3 \end{aligned}$$

Now we can use the fact that the point (-2, -1) lies on our graph in order to help us find a, because if (-2, -1) lies on the graph, that means that f(-2) = -1. Therefore, we can plug in:

$$f(-2) = -1 \quad \text{plug in } x = -2 \ \& \ f(x) = -1$$

$$\begin{aligned} -1 &= a(-2+1)^2 - 3 \\ -1 &= a(-1)^2 - 3 \\ -1 &= a - 3 \\ &\underline{a = 2} \end{aligned}$$

We will be able to find a. Plugging this value of a into our equation, and transform our answer in standard form gives us:

$$\begin{aligned} f(x) &= a(x-h)^2 + K \\ f(x) &= a(x-(-1))^2 + (-3) \\ f(x) &= a(x+1)^2 - 3 \\ f(x) &= 2(x+1)^2 - 3 \\ f(x) &= 2(x^2 + 2x + 1) - 3 \\ f(x) &= 2x^2 + 4x + 2 - 3 \\ &\underline{f(x) = 2x^2 + 4x - 1} \end{aligned}$$

2 Converting a Quadratic Function from Standard to Vertex Form

For example, let us convert this function here

$$f(x) = x^2 + 2x - 2$$

to the vertex form $f(x) = a * (x - h)^2 + k$, and then we will give the vertex of its graph. Now, our function here is written in standard form, and in order to convert a quadratic function written in standard form into this vertex form here, we complete the square.

The first step in completing the square is to make sure that the coefficient of this square term is 1. We will take ½ as the coefficients of x in our case.

$$\begin{aligned} f(x) &= x^2 + 2x - 2 \quad \text{with } \frac{1}{2}(2) = 1 \\ f(x) &= x^2 + 2x + 1 - 1 - 2 \\ f(x) &= (x+1)^2 - 3 \quad \textit{with } x - (-1) \\ h &= -1, k = 3 \\ &\underline{V = (-1, -3)} \end{aligned}$$

Therefore, we have written our function in vertex form. When we write our quadratic function in this vertex form, then h, k is the vertex of its graph.

Let us see another example.

$$f(x) = -3x^2 + 12x - 11$$

We are going to convert this to vertex form, and then give the vertex of its graph. Again, we will begin in the same way by completing the square.

Now we notice that the coefficient of this square term is ≠ 1, and when completing this square, what we need to do then in this case is factor that number out of both of these two x-terms.

$$\begin{aligned} f(x) &= -3x^2 + 12x - 11 \\ f(x) &= -3(x^2 - 4x + 4-4) - 11 \qquad \text{with } \frac{1}{2}(-4) = -2 \\ f(x) &= -3(x^2 - 4x + 4) + (-3)(-4) - 11 \\ f(x) &= -3(x-2)^2 + 1 \\ h &= 2, k = 1 \\ &\underline{V = (2.1)} \end{aligned}$$

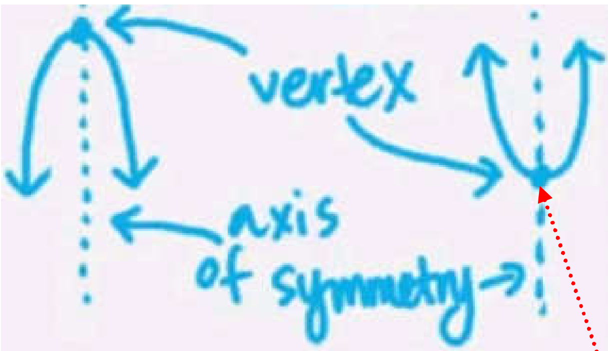
Our equation is now in vertex form, and again comparing it to this form, which means that our vertex is at 2, 1.

3 Graphing a Parabola

For example, let us sketch the graph of the quadratic function here.

$$f(x) = x^2 - 8x + 15$$

Well, the graph of a quadratic function is what we call a parabola. An important point on a parabola is called the vertex. It is the point where the parabola turns around, therefore, this would be called the vertex here, or this would be considered the vertex.



The vertex is also the only point on the parabola that lies on what we call the axis of symmetry. This is the line about which the parabola is symmetrical. That is the graph is a mirror image on either side of this axis of symmetry.

Now, in general we can graph a parabola by plotting its vertex and a few points on either side of it. How do we find the vertex? Well, we have the following vertex form for the equation of a quadratic function.

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the vertex.

In addition, if $a > 0$, then the parabola will open upward like the second figure here. If $a < 0$, then the parabola will open downward like the first figure. How do we get a quadratic function that is in standard form, like ours here, into vertex form? Well, we complete the square! Let us do that.

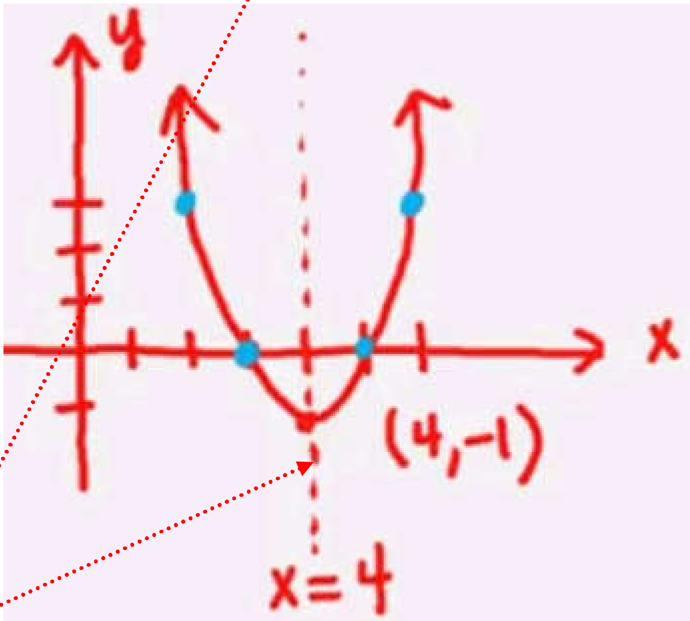
$$\begin{aligned} f(x) &= x^2 - 8x + 15 \text{ with } \frac{1}{2}(-8) = -4 \\ &= x^2 - 8x + 16 - 16 + 15 \\ &= (x-4)^2 - 1 \\ a &= 1, h = 4, k = -1 \\ &\underline{V = (4.-1)} \end{aligned}$$

Since $a > 0$, our parabola will be opening upward, and our vertex is (4, -1).

Now let us find some other points in our parabola.

x	y
2	3
3	0
5	0
6	3

Let us plot these points, and our parabola will look like this.



This is the sketch that we are looking for. However, let us notice a few things. Let us draw our line of symmetry here, which is $x = 4$, and doing this we can see that this graph is symmetric about that axis, or the mirror image on either side of it. Therefore, looking back over here in the table, we really only needed to determine these two points here, because we get the other two by symmetry, do we not? Using this axis of symmetry is very useful when sketching parabolas.

Let us look at graphing a parabola. For example, let us sketch the graph of this quadratic function:

$$f(x) = -3x^2 + 18x - 22$$

Then we will find the maximum or minimum value of f, as well as its domain and range, and the intervals where it is increasing or decreasing.

To sketch the graph of a quadratic function, or a parabola, we plot the vertex, and a few points on either side of it. Our quadratic function here is written in standard form, and what we need to do is convert it to vertex form.

$$\begin{aligned} f(x) &= -3x^2 + 18x - 22 \\ &= -3(x^2 - 6x + 9 - 9) - 22 \qquad \text{with } \frac{1}{2}(-6) = -3 \\ &= -3(x-3)^2 + 27 - 22 \\ &= -3(x-3)^2 + 5 \end{aligned}$$

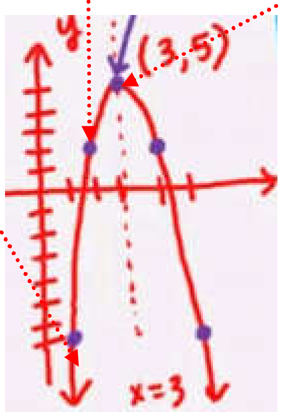
$$a = -3, h = 3, k = 5$$

$$\underline{V = (3,5)}$$

Since $a < 0$ our parabola will be opening downward, and our vertex is at (3, 5). Now let us find some other points in our parabola.

x	y
1	-7
2	2

Therefore, 1, -7 will be here, 2, 2 will be up here, and our vertex is here. Now, because of symmetry, we know that (4, 2) also has to lie on our graph, as well as (5, -7). Therefore, our parabola will look like this.



It is very helpful to use this axis of symmetry to get the other points. That is, by knowing the point 2, 2 lies on our graph, we can conclude that 4, 2 do as well. In addition, that we knew the point 1, -7 lay on our graph, then we can conclude that 5, -7 does as well.

Here we have the graph of our quadratic function, or our parabola, but we are still asked to find the maximum or minimum values of *f*, its domain, range, and intervals of increase or decrease. Now, because this parabola is opening downward, it will have a maximum, not a minimum, and the maximum will occur at the vertex. That is the maximum value of our function is the y-coordinate of that vertex.

$$\begin{aligned} \text{Maximum} &= 5 \\ D &= (-\infty, \infty) \\ R &= (-\infty, 5] \\ \text{increase}&: (-\infty, 3) \\ \text{decrease}&: (3, \infty) \end{aligned}$$

4 Maximum/Minimum Quadratic Function Word Problems

A ball is thrown vertically upward, after t [s] its height, h [feet], is given by this function here.

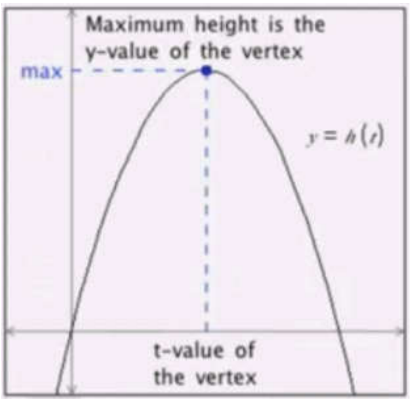
$$h(t) = 64t - 16t^2$$

We are going to find how long it is going to take for this ball to reach its maximum height.

Well, the ball's height here, $h(t)$, is a quadratic function that is throughout this a parabola, and sets the coefficient of the square term, namely -16, is < 0, this parabola will be opening downward. This means that the ball will reach its maximum height at the vertex of this parabola.

To find the time that the ball will reach its maximum height, we need to find the t-value of the vertex. That is we need to find the value for which this ball reaches a maximum. There are a few different ways to find this vertex.

We can both complete the square, and put the equation in vertex form, but since we are only interested in how long it will take to reach its maximum height, which means we only need to find the t-value of the vertex. We can do that by using the fact that the vertex of a quadratic equation ($at^2 + bt + c$) is at $t = {}^{-b}/_{2a}$, which actually is a direct result from completing this square.



Looking at our equation, we see that a = -16, and b = 64, the coefficient of t. Therefore, the t-value of the vertex is 2. That is our answer then, that after t = 2 s this ball will reach its maximum height.

5 Quadratic Inequalities

For example, let us solve the following inequality.

$$x^2 - 4x > 5 \quad > \text{is strict}$$

Now, when we solve quadratic inequalities, we start in the same way as we do when we solve quadratic equations, namely bringing everything to one side and getting zero on the other. Then we can factor the left-hand side. Therefore, this gives us:

$$x^2 - 4x > 5$$
$$x^2\text{-}4x\text{-}5 > 0$$
$$(x\text{-}5)(x+1) > 0$$

Now, when solving these types of inequalities, there are a few different approaches. The first approach is what we call a sign analysis. Remember, if we have a product $A * B > 0$, what does that mean? That means that either A and B are *both* +, or A and B are *both* -. In other words:

$$A * B > 0$$
$$[A > 0 \& B > 0]$$
$$or$$
$$[A < 0 \& B < 0]$$

Let us apply that here. We have a product of factors > 0, which means:

$$[(x-5) > 0 \& (x+1) > 0] \quad \text{or} \quad [(x-5) < 0 \& (x+1) < 0]$$
$$[x > 5 \& x > -1] \quad \text{or} \quad [x < 5 \& x < -1]$$

Now we can consolidate this in to just $x > 5$ and $x < -1$.

Combining these we get $x < -1$ or $x > 5$. Notice that these values are strict, because our original inequality is a strict inequality. We are asked to put our answer in interval notation. Therefore, our answer has to be:

$$\underline{(-\infty, -1) \cup (5, \infty)}$$

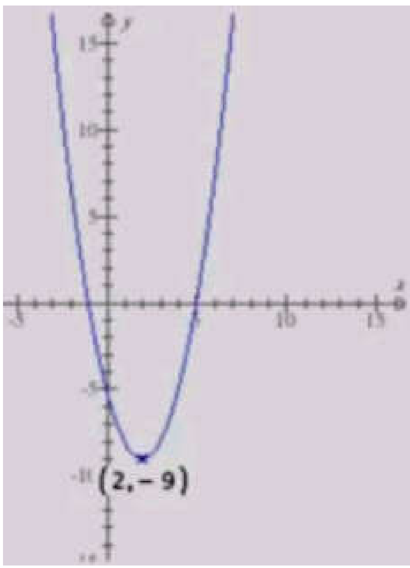
What is about the other approach? The second approach is what we call a graphical approach. So we still start with the same inequality, and we begin in the same way.

$$x^2 - 4x > 5$$
$$x^2\text{-}4x\text{-}5 > 0$$

Now, let:

$$y = x^2 - 4x - 5$$

Therefore, to solve this we need to find x-values that make $y > 0$, which we can do by thinking of the graph of y.



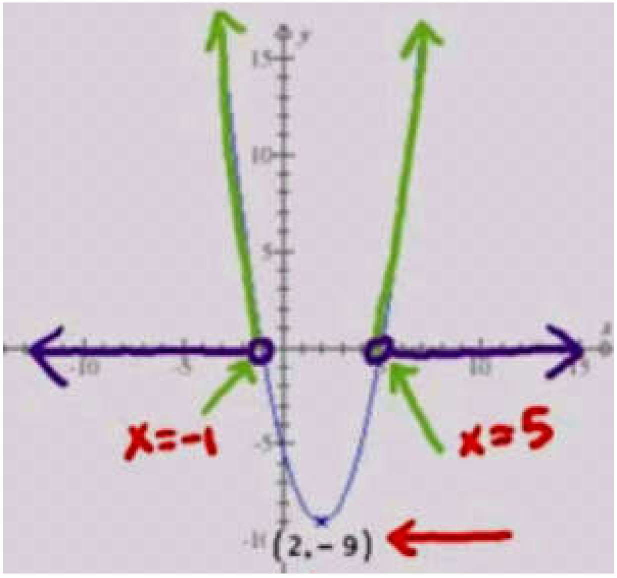
The graph of y is a parabola, is it not? In addition, because the leading coefficient here, namely $a > 0$, we have the parabola opening upward, and we found the vertex by using the formula:

$$x = \frac{-b}{2a}$$
$$x = \frac{-(-4)}{2(1)}$$
$$= 2$$

Then we can find the y-coordinate of the vertex by plugging that value 2 into the equation:

$$y = x^2 - 4x - 5$$
$$y = (2)^2 - 4(2) - 5$$
$$= -9$$

This is how we get this vertex. Now remember, we want to find the x-values that make $y > 0$.



$y > 0$ is above the x-axis here on these green pieces. Therefore, what we need to do is to determine what x-values will land us on those green pieces. This means, we need to find these x-intercepts, and we can do that by factoring, and the x-intercepts are the values where this is zero.

$$y = x^2 - 4x - 5 \rightarrow$$
$$x\text{-}5x + 1 = 0$$
$$x = 5, -1$$

We know $y = 0$ at those values, but we are only interested in the x-values that make y strictly > 0. Therefore, we need to exclude those values in our answer. As long as $x > 5$ or $x < -1$ we will land on the green pieces of the parabola. Therefore, our answer is the same as it was in the first approach.

6 Finding the Inverse of a Function

For example, let:

$$f(x) = 1 - 4x$$

Let us find, $f^{-1}(x)$ if it exists.

Remember that the inverse of a function is defined as follows.

$$y = f^{-1}(x) \Leftrightarrow f(y) = x$$

Remember; be very careful with this notation. This does mean:

$$f^{-1} \neq \frac{1}{f}$$

It is just the notation for the inverse function.

Notice what we are doing here is we are switching the roles of *x* and *y*, and then solving for *y*, and that will give us $f^{1}(x)$.

$$\begin{aligned} f(x) &= 1\text{-}4x \\ y &= 1\text{-}4x \\ x &= 1\text{-}4y \\ 4y &= 1\text{-}x \\ y &= \frac{1\text{-}x}{4} \\ y &= \frac{1}{4} - \frac{x}{4} \\ \underline{f^{-1}(x)} &\underline{= \frac{1}{4} - \frac{x}{4}} \end{aligned}$$

All right, let us look at another example.

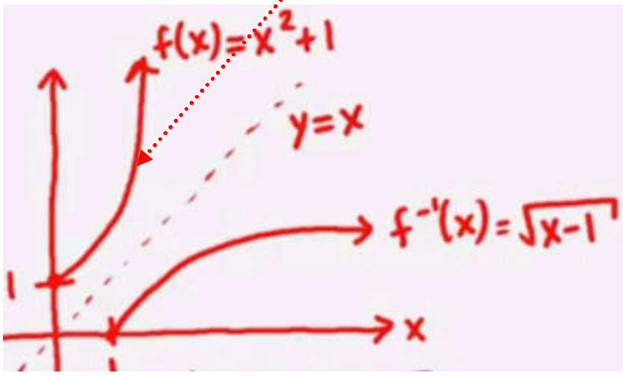
$$f(x) = x^2 + 1$$

Let us find $f^{-1}(x)$ if it exists.

$$\begin{aligned} f(x) &= x^2 + 1 \\ y &= x^2 + 1 \\ x &= y^2 + 1 \\ y^2 &= x\text{-}1 \\ y &= \pm\sqrt{x\text{-}1} \\ &\underline{f^{-1}(x)\text{ does not exist}} \end{aligned}$$

Which means, we were not uniquely able to solve for *y*, because for any input x we would get two possible y-values, the + and the – square root. This means, that $f^{1}(x)$ does not exist. This should not surprise you, if you think of the graph of this function. It is a parabola with vertex at (0, 1) opening upward. Which is not a 1 : 1-function, is it?

Because by the horizontal line test, if we pass a horizontal line through this graph, it going to intersect it in more than 1 spot. If the original function *f* is not 1 : 1, then $f^{-1}(x)$ will not exist as a function. However, if we restrict dom(*f*) = [0, ∞), then we are only looking at this part of the parabola.



On that restricted domain, *f* is 1 : 1. Therefore, *f*, on the restricted domain, will have an inverse. That is, if we restrict the domain of *f*, we could have also restricted it from -∞ to 0, but let us just restricts it from 0 to ∞. Then *f* will be 1 : 1, and therefore, f^{1} will exist. If we look at the graph of *f*(*x*) on the restricted domain, and its inverse will exist.

However, what are we going to choose for $f^{1}(x)$, the + or the -? We are going to choose the +, because remember that the range of the inverse is equal to the domain of the original function. If we are restricting that domain 0 to ∞, this has to be the range of the inverse, and so the output, namely *y*, has to be +. That is on this restricted domain:

$$\underline{f^{-1}(x) = \sqrt{x-1}}$$

Moreover, look at these graphs. Are they not symmetric about the line *y* = *x*? That will always be true for a function and its inverse. The graphs will always be symmetric about the line *y* = *x*. You should graph the two lines from example number one, you will see that the graphs are symmetric about the line *y* = *x*.

7 Inverse of a Function

For example, let us find the inverse of this 1 : 1-function *f*, and then we will state its domain and range.

$$f(x) = \frac{x+3}{x-4}$$

Let us find $f^{1}(x)$ if it exists. The first thing we do, when we are finding an inverse of a function is, in the definition of that function we replace *f*(*x*) by *y*. Then we interchange the rules of *x* and *y*. Now we want to manipulate the equation in order to solve for *y*. When we solve for *y*, *y* will be equal to *f* inverse of *f*.

$$\begin{aligned} f(x) &= \frac{x+3}{x-4} \\ y &= \frac{x+3}{x-4} \\ x &= \frac{y+3}{y-4} \\ x(y-4) &= y+3 \\ xy-4x &= y+3 \\ xy-y &= 3+4x \\ y(x-1) &= 3+4x \\ y &= \frac{3+4x}{x-1} \\ \underline{f^{-1}x} &\underline{= \frac{3+4x}{x-1}} \end{aligned}$$

Here is our inverse. However, we also are asked to find its domain and range. Let us start with the domain. Looking here at f^{-1}, the only issue would be if $x = 1$, because then we would be dividing by zero, therefore, we need to exclude this value. What is about the range? Now, this is where we can use the fact that the range of the inverse is the domain of the original function. That is this is equal to the domain of *f*. However, what is this domain? Looking back up at our original function, the only issue would be if $x = 4$, because then we would be dividing by zero. That is we need to exclude this value in the domain of *f*.

$$\begin{aligned} \text{Dom}\left(f^{-1}\right) &= (-\infty,0) \cup (1,\infty) \\ \text{Range}\left(f^{-1}\right) &= \text{Dom}(f) = (-\infty,4) \cup (4,\infty) \end{aligned}$$

It is very useful to be able to look at the domain of the original function to determine the range of the inverse.

8 Verifying Inverse Functions

For example, let us determine whether

$$f(x) = 5-2x \text{ and } \quad g(x) = \frac{5}{2} - \frac{x}{2}$$

are the inverse of each other. We have the following fact. The two functions *f* and *g* are inverses of each other, if and only if both of the following compositions hold.

$$\begin{aligned} f(g(x)) &= x \quad (\text{for all x in the domain of } g) \\ g(f(x)) &= x \quad (\text{for all x in the domain of } f) \end{aligned}$$

That is the composition in either direction gets us back to *x*. Let us compute this for our situation here.

$$\begin{aligned} f(g(x)) &= f\left(\frac{5}{2} - \frac{x}{2}\right) \\ &= 5 - 2\left(\frac{5}{2} - \frac{x}{2}\right) \\ &= 5 - 2\left(\frac{5}{2}\right) - 2\left(-\frac{x}{2}\right) \\ &= 5 - 5 + \frac{2x}{2} \\ &\underline{= x} \end{aligned}$$

This first composition property is satisfied. However, we have to make sure that the second one is also satisfied.

$$\begin{aligned} g(f(x)) &= g(5-2x) \\ &= \frac{5}{2} - \frac{(5-2x)}{2} \\ &= \frac{5}{2} - \frac{5}{2} + \frac{2x}{2} \\ &= \underline{x} \end{aligned}$$

The second composition property is satisfied as well. Therefore, yes, *f* and *g* are inverses of each other. That is, we can say that $f = g^{-1}$, and $g = f^{-1}$.

All right, let us look at another example. Let us determine whether these two functions are inverses of each other.

$$f(x) = \frac{4}{x} \qquad \text{and} \qquad g(x) = -\frac{4}{x}$$

Again, we are going to compute both of these compositions, and see if they yield *x*.

$$\begin{aligned} f(g(x)) &= f\left(-\frac{4}{x}\right) \\ &= \frac{4}{\frac{-4}{x}} \\ &= 4 * \frac{x}{-4} \\ &= \underline{-x \neq x} \end{aligned}$$

It is not equal to *x*; therefore, this is not satisfied. These will not be inverses of each other. However, let us, for the fun of it, compute the composition of it in the other direction.

$$\begin{aligned} g(f(x)) &= g\left(\frac{4}{x}\right) \\ &= \frac{-4}{\frac{4}{x}} \\ &= -4 * \frac{x}{4} \\ &= \underline{-x \neq x} \end{aligned}$$

This again, is ≠ x. Therefore, the second composition down here is also not satisfied.

Now be careful here! Some students will think that, because *g* composed with *f*, and *f* composed with *g*, are equal to one another. That this functions would be inverses of each other, but they are not, because not only do these compositions have to be equal to one another, they also have to be equal to *x*, not -*x*.

Therefore, our answer here then would be no, they are not inverses.

9 Polynomial Long Division

For example:

$$\frac{x^2 + 9x + 19}{x + 4}$$

Now we begin dividing polynomials in a similar way as we do when we divide numbers, but we start by looking at the leading terms, and ask ourselves, *x* times what is = x^2. This would be *x*, would it not? Because $x^2 / x = x$. Therefore, that is the first term in our quotient *x*. Now we need to multiply *x* by the entire divisor $x + 4$, and then, just like when we divide numbers, we now subtract $x^2 - x^2 = 0$, $9x - 4x = 5x$, and we still have +19. Now are we done? We are not! We continue until the degree is smaller than the degree of the divisor.

Again, we look at the leading terms, x and $5x$, and ask ourselves, x times what is = $5x$, and this would be 5, would it not? Because $5x / x = 5$. Therefore, that is the next term in our quotient. Now we have +5, and now we multiply 5 by the entire divisor $x + 4$, which gives us $5x + 20$. Again, we will subtract $5x - 5x = 0$, and 19 – 20 = -1, and now the degree of -1 is 0, which is smaller than the degree of the divisor; we are done.

Well how can we represent our answer?

$$\frac{x^2+9x+19}{x+4}$$

$$\begin{array}{rl} & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; x+5 \\ x+4 \!\!\!\!& \big)\; x^2+9x+19 \\ - & \;\;\; x^2+4x \\ \hline & \;\;\;\;\;\;\; 5x+19 \\ - & \;\;\;\;\;\;\; 5x+20 \\ \hline & \;\;\;\;\;\;\;\;\;\;\;\; -1 \end{array}$$

$$\frac{x^2+9x+19}{x+4} = (x+5) + \frac{-1}{x+4}$$
$$= \underline{(x+5)(x+4) - 1}$$

These are two nice ways of representing our answer. In this last form here, we can actually check that we have done this division correctly.

$$(x+5)(x+4) - 1 = x^2 + 4x + 5x + 20 - 1$$
$$= \underline{x^2 + 9x + 19}$$

All right, let us look at another example.

$$\frac{7 + 15x + 4x^4 - 11x^2}{2x^2 + 3x - 2}$$

The first thing to notice here is that our dividend is not written in standard form. Standard form would be

$$\frac{4x^4 - 11x^2 + 15x + 7}{2x^2 + 3x - 2}$$

The other thing to notice, is also, there is no x^3-term. Let us add a placeholder term with a coefficient of zero.

$$4x^4 - 11x^2 + 15x + 7 = 4x^4 + 0x^3 - 11x^2 + 15x + 7$$

We are going to ask ourselves, $2x^2$ times what is = $4x^4$. Is that not $2x^2$? Because $4x^4 / 2x^2 = 2x^2$. All right, now we have to ask ourselves, how many times $2x^2$ goes into $-6x^2$? That would be -3x, would not it? Because $-6x^3 / 2x^2 = -3x$, which is the next term in our quotient. Therefore, we are ready to divide now.

$$\frac{4x^4 - 11x^2 + 15x + 7}{2x^2 + 3x - 2}$$

$$\begin{array}{rl} & \; 2x^2 - 3x + 1 \\ 2x^2+3x-2 \!\!\!\!& \big)\; 4x^4 + 0x^3 - 11x^2 + 15x + 7 \\ - & \;\;\; 4x^4 + 6x^3 - 4x^2 \\ \hline & \;\;\;\;\;\;\;\; -6x^3 - 7x^2 + 15x + 7 \\ - & \;\;\;\;\;\;\;\; -6x^3 - 9x^2 + 6x \\ \hline & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; 2x^2 + 9x + 7 \\ - & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; 2x^2 + 3x - 2 \\ \hline & \; 6x + 9 \end{array}$$

$$\frac{4x^4 - 11x^2 + 15x + 7}{2x^2 + 3x - 2} = \underline{(2x^2 - 3x + 1) + \frac{(6x+9)}{2x^2 + 3x - 2}}$$

10 Synthetic Division

For example, let us use synthetic division:

$$P(x) = \frac{2x^3 - 9x^2 + 6x - 7}{x - 4}$$

Now, synthetic division is a shortcut used to divide a polynomial by x - r. Notice here, this is a degree 1 polynomial. So synthetic division can only be used when the divisor is a degree 1 polynomial.

This technique comes from looking at the coefficients of *P*(*x*) and the value of *r*. Now, what are the coefficients of *P*(*x*) here? Well, let us put them under a division sign.

It is imperative that our polynomial is in standard form before we read off these coefficients, but, moreover, if there were any missing powers of *x* here, we would have to write zero for that power as a place holder. Now, what is our *r* here? Our *r* is 4. Let us put the 4 to the left of the division symbol.

$$P(x) = \frac{2x^3 - 9x^2 + 6x - 7}{x - 4}$$

$$x - 4\overline{)2 \quad -9 \quad 6 \quad -7}$$

Now that these numbers are set up, the first step in synthetic division is to drop the first coefficient of *P*, therefore, we will drop the 2. The next step is that we multiply the 4 and the 2, and then we add; -9 + 8 = -1. Now we continue this pattern. We multiply the 4 * -1, which gives us -4. Again, we are going to add, which gives us 2, and then we multiply, which gives us 8 and then we add.

$$P(x) = \frac{2x^3 - 9x^2 + 6x - 7}{x - 4}$$

$$\begin{array}{r|rrrrl} 4 & 2 & -9 & 6 & -7 & \\ & 0 & 8 & -4 & 8 & \text{Multiplying} \\ & 2 & -1 & 2 & 1 & \text{Adding} \end{array}$$

Now, what are these numbers down there? These first three numbers are the coefficients of the quotient of the division, and this last number is the remainder of the division. Moreover, the degree of this quotient is 1 less than the degree of our dividend, *P*(*x*). This is because our divisor *x* - r is a degree 1 polynomial. Since our dividend, *P*(*x*), is a degree 3 polynomial, then our quotient will be a degree 2 polynomial. Therefore, our quotient, which we will call *Q*(*x*), is equal to a degree 2 polynomial with these coefficients.

$$Q(x) = 2x^2 - x + 2, R(x) = 1$$

Therefore, by the division algorithm *P*(*x*) / *x* - *r* is = *Q*(*x*) which:

$$\frac{P(x)}{x - r} = \left(2x^2 - x + 2\right) + \frac{1}{x - 4}$$

Let us actually take a quick look at the long division, that you can see why this technique works.

$$\begin{array}{r} 2x^2 - x + 2 \\ x - 4\overline{)2x^3 - 9x^2 + 6x - 7} \\ \underline{-2x^3 - 8x^2 \qquad\qquad} \\ -x^2 + 6x - 7 \\ \underline{-x^2 + 4x \quad\;\;} \\ 2x - 7 \\ \underline{-2x + 8} \\ \underline{\underline{1}} \end{array}$$

Now, sometimes students get confused that we subtract here, but we add with synthetic division. Notice here we have this - in front of the 4, whereas with synthetic division we do not. Therefore, subtracting then here will give us the same result as adding when we use synthetic division.

All right, let us look at one more example. Let us use synthetic division:

$$P(x) = \frac{1 + 4x^2 - 2x^4 - 7x^3}{x + 4}$$

Now, the first thing to notice about *P*(*x*) here is that it is not written in standard form. Therefore, let us do that, let us write it in standard form.

$$P(x) = \frac{-2x^4 - 7x^3 + 4x^2 + 1}{x + 4}$$

Notice here that there is no x-term, and with synthetic division, it is very important that we have every power of *x*. That is, before reading off the coefficients of *P*, we need to write *P*(*x*) as follows:

$$P(x) = \frac{-2x^4 - 7x^3 + 4x^2 + 0x + 1}{x + 4}$$

We are ready to read of the coefficients of *P*. They are -2, -7, 4, 0, and 1. However, what is *r*? We are dividing *P* by x + 4, and we can think of x + 4 as:

$$x + 4 = x - (-4)$$

Therefore, this is *r* here. Now:

$$\begin{array}{rrrrrrl} -4 & \sqrt{-2} & -7 & 4 & 0 & 1 & \\ & & 8 & -4 & 0 & 0 & \text{Multiplying} \\ & -2 & 1 & 0 & 0 & 1 & \text{Adding} \end{array}$$

Remember, these are the coefficients of our quotient, but, moreover, our quotient is 1 less in degree than the dividend. This means:

$$Q(x) = -2x^3 + x^2$$

In addition, this is our remainder, *r*(*x*). Therefore, by the division algorithm, we have:

$$\frac{P(x)}{x - r} = Q(x) + \frac{R(x)}{x - r}$$

$$\frac{1 + 4x^2 - 2x^4 - 7x^3}{x + 4} = -2x^3 + x^2 + \frac{1}{x + 4}$$

11 Properties of Polynomials

For example let us state the left and right behavior. Find the maximum number of 0's, and the maximum number of local extrema for this polynomial here:

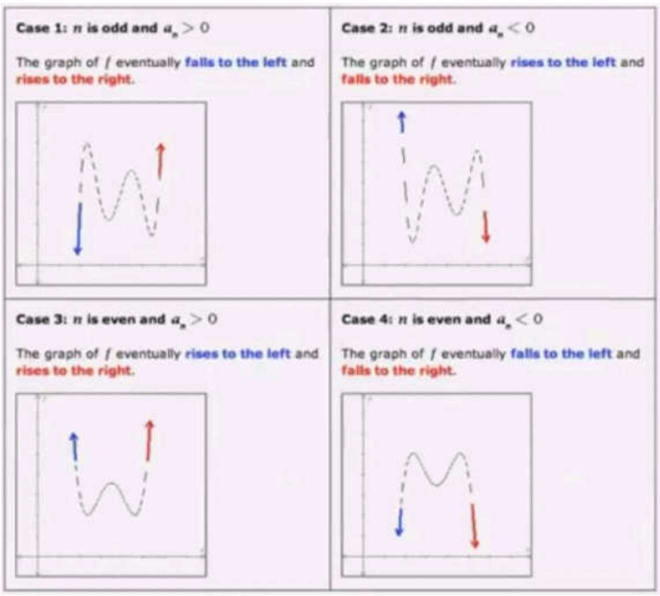
$$f(x) = 6x^4 - 7x^3 + 4x^2 + 6x$$

Let us begin by looking at the left and right behavior, or the end behavior, of this polynomial. Now a polynomial function is of the following form.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

This here is called the degree of the polynomial, and this here is called the leading coefficient, which we are assuming is ≠ 0.

Now, the graph of any polynomial function *f* will eventually rise or fall without bound, and to determine what happens, we look at this degree, and the leading coefficient, namely we have the following 4 cases.



Looking at our polynomial, we see that the degree is 4, and that the leading coefficient is 6. That is n = 4, which is even, and a_n = 6, which is > 0. Therefore, we are in case 3. This means that we can write our answer up here that the graph of *f* eventually rises to the left and rises to the right.

Now, it still remains to find the maximum number of zeros, or x-intercepts, and the maximum number of local extrema. Again, the degree governs that. We have the following two cases.

A polynomial of degree *n* with real coefficients has at most *n* real zeros, and at most *n-1* local extrema or turning points. Therefore, for our polynomial we have n = 4. This means, coming up back here, the maximum number of zeros then will be 4, and the maximum of local extrema will be 3.

All right, let us look at one more. Let us state the left and right behavior of this polynomial here, as well as find the maximum number of zeros, and the maximum number of local extrema.

$$f(x) = -2(x-2)^2(x+4)$$

Let us start again with the left and right behavior. Well, what are the degree and the leading coefficient of our polynomial? If we multiplied it all out, we would have:

$$f(x) = -2x^3 + ... \qquad \text{a bunch of other stuff}$$

We are going to have a degree 2 polynomial, which then multiplying a degree 1 polynomial, which would give us a degree 3 polynomial, with a leading coefficient of -2. That means n = 3, which is odd, and a_n = -2, which is negative. This means we are in case 2, are we not? Therefore, the graph of *f* eventually rises to the left and falls to the right.

Now, let us take the maximum number of zeros as well as the maximum number of local extrema. Recall from our last example that a polynomial of degree ***n*** with real coefficients has at most ***n*** real zeros and at most ***n-1*** local extrema. Here n = 3, which means, coming back up here, the maximum number of zeros then would be 3, and the maximum number of local extrema would be 2.

12 Remainder Theorem

For example, let us use the remainder theorem to find *P*(*1*), where:

$$P(x) = 2x^3 - 3x^2 - 9$$

The remainder theorem states the following. If a polynomial *P*(*x*) is divided by (*x* - *c*) then the remainder of that division is *P*(*c*). Let us think about why this is true.

$$\frac{P(x)}{x-c} = (x-c)Q(x) + R$$

divisor quotient remainder

The reason that the remainder is a constant is, because it has to be 1 less in degree than this divider. This is of degree 1, therefore, *R* would have to be of degree 0, or a constant.

Now let us evaluate *P*(*c*):

$$\begin{aligned} P(c) &= (c-c)Q(c) + R \\ P(c) &= 0 * Q(c) + R \\ P(c) &= R \end{aligned}$$

$$\begin{array}{r} 2x^2 - x - 1 \\ x-1 \overline{\smash{)}\, 2x^3 - 3x^2 + 0x - 9} \\ \underline{-2x^3 + 2x^2} \\ -x^2 + 0x - 9 \\ \underline{x^2 - \ \ x} \\ -x - 9 \\ \underline{x - 1} \\ -10 \leftarrow \text{R} \end{array}$$

Therefore, *P*(1) = -10, which is our answer.

Now we can check this answer by plugging this value into our polynomial *P*.

$$\begin{aligned} P(1) &= 2(1)^3 - 3(1)^2 - 9 \\ &= 2 - 3 - 9 \\ &= \underline{-10} \end{aligned}$$

Let us look at another example. Again, let us use the remainder theorem to find *P*(-3):

$$P(x) = x^4 + 2x^3 - 4x^2 + 5$$

Again in order to find *P*(c), we divide *P*/(*x* - *c*), where here in this case, *c* = -3. We would need to divide our polynomial by (*x* — 3) or (*x* + 3).

Again, in order to perform this division, we could use either long division or synthetic division. Let us use synthetic division here. Therefore, we write our *c*, which is -3, and then we put all the coefficients of *P*, namely 1, 2, -4,…. Now be careful here. Remember, it is mandatory that we hold the place of the *x* and write + 0x. Therefore, we have a 0, and then 5, and then we drop the 1.

$$\begin{array}{r|rrrrr} -3 & 1 & 2 & -4 & 0 & 5 \\ & & -3 & 3 & 3 & -9 \\ \hline & 1 & -1 & -1 & 3 & -4 \end{array}$$

Now remember with synthetic division, this last number here is our remainder. Therefore, that is *P*(-3) by this theorem. Our answer then is that *P*(-3) = -4. Again, we can check our answer by plugging -3 into our polynomial.

$$\begin{aligned} P(-3) &= (-3)^4 + 2(-3)^3 - 4(-3)^2 + 5 \\ &= 81 - 54 - 36 + 5 \\ &= \underline{-4} \end{aligned}$$

13 Factor Theorem

For example, let us determine whether (x - 3) is a factor of:

$$P(x) = 2x^3 - 5x^2 - x - 6$$

We can use the factor theorem to help us out.

Factor Theorem

A polynomial $P(x)$ has a factor $x - r$ if and only if $P(r) = 0$.

We want to use this to determine whether (*x* - 3) is a factor of the polynomial. Therefore, our *r* = 3, and we want to compute

$$\begin{aligned} P(3) &= 2(3)^3 - 5(3)^2 - 3 - 6 \\ &= 2(27) - 5(9) - 3 - 6 \\ &= 54 - 45 - 3 - 6 \\ &= \underline{0} \end{aligned}$$

Since *P*(3) = 0, (*x* - 3) is a factor of *P*.

Let us see another example. Let us determine whether (*x* + 2) is a factor of:

$$P(x) = -2x^4 - 2x^3 + 6x^2 - 5$$

Again, we will use the factor theorem, which states that (*x* - r) will be a factor, if *P*(0) = *r*. We want to determine whether (*x* + 2) is a factor of *P*(x).

However, what is our *r* here? (*x* + 2) is really (*x* — 2). Be careful here, our *r* = -2. Therefore, we will plug in -2 everywhere we see an *x*, and if we get 0 as an answer then, yes, (*x* + 2) will be a factor of *P*.

$$\begin{aligned} P(-2) &= -2(-2)^4 - 2(-2)^3 + 6(-2)^2 - 5 \\ &= -2(16) - 2(-8) + 6(4) - 5 \\ &= -32 + 16 + 24 - 5 \\ &\underline{= 3} \end{aligned}$$

Since $P(-2) \neq 0$, $(x + 2)$ is not a factor of P.

Exponential & Logarithmic Equations

1 Introduction to Exponential & Logarithmic Functions

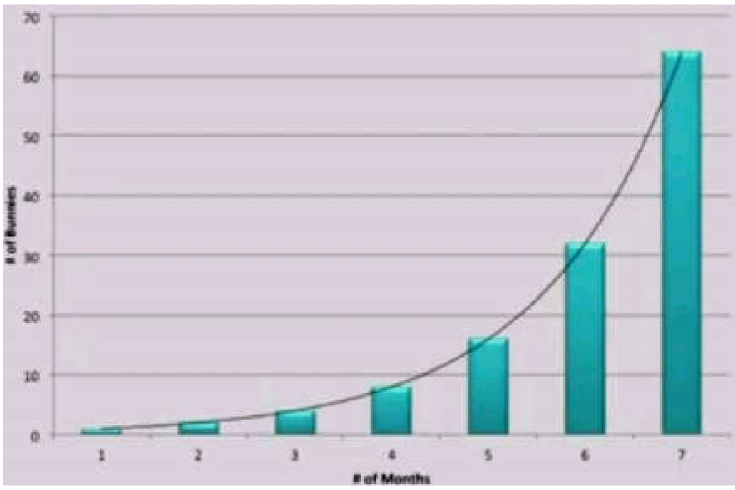
You have already spent a great deal of time learning about functions, but these two functions are so special, that they are worth their own unit. These functions get used a lot in applications.

Let us start with an application to see why we need an exponential function and a logarithmic function, and why the ones you have learned so far would not suffice. Let us begin by looking at an example to understand why we need the exponential and logarithm functions.

Suppose we have a bunny, and if this bunny has a bunny every month, well, after 1 month we would have 1 bunny. After 2 months, he would have a bunny, and we would have 2 bunnies. After 3 months, each of those 2 bunnies would have a bunny, and we would have 4 bunnies. If each of those 4 bunnies had a bunny, we would have 8 bunnies, a.s.o. Moreover, eventually we would have a ton of bunnies.

Suppose we track the bunny's population over time. What if we wanted to know how many bunnies there were after a month, after 2 months, after 3 months, after a year, after 5 years, after a century?

To do these sorts of calculations, we would need to have some sort of function, which captures the doubling behavior of the bunny population. Therefore, the function we are going to look at is something called the exponential function. We have a graph, and in this graph, we see the population of the bunnies for several iterations.



We started with 1 bunny, then we had 2, then we had 4, a.s.o. What kind of function matches this behavior?

Well, if you look back, think about the catalog of functions you already know, you know linear functions, polynomial functions, rational functions, and you try to think about this curve over here.

What does that look like? This curve does not look like any of the curves you are familiar with. You might guess a parabola, because it seems to be curving up. However, that would not work for this population.

If we want to match this with a curve, it would have to look something like this, and that curve is getting steeper and steeper, because this population of bunnies is getting large very quickly. That brings us to the need for the exponential function.

In John Napier's book is a wonderful description of logarithms. He described how the logarithm was such a useful tool to help do calculations for celestial mechanics. This book was the first one of its time to describe, how a logarithm could be used to turn hard multiplications into simple additions. He also gave a table of logarithmic values, which was used for hundreds of years afterwards.

Another mathematician, Leonard Euler, played a large role in the development of the exponential and logarithmic functions. Leonard Euler was actually the first person to use the notation for a function $f(x)$. Whenever you see the $f(x)$ that was due to Euler. Euler was a very prolific mathematician. He wrote over 800 published papers in his life time, and his works would complete 90 volumes. In one of Euler's works he looked at a constant that came up from numerous calculations, and it is since been known as Euler's constant, or Euler's number.

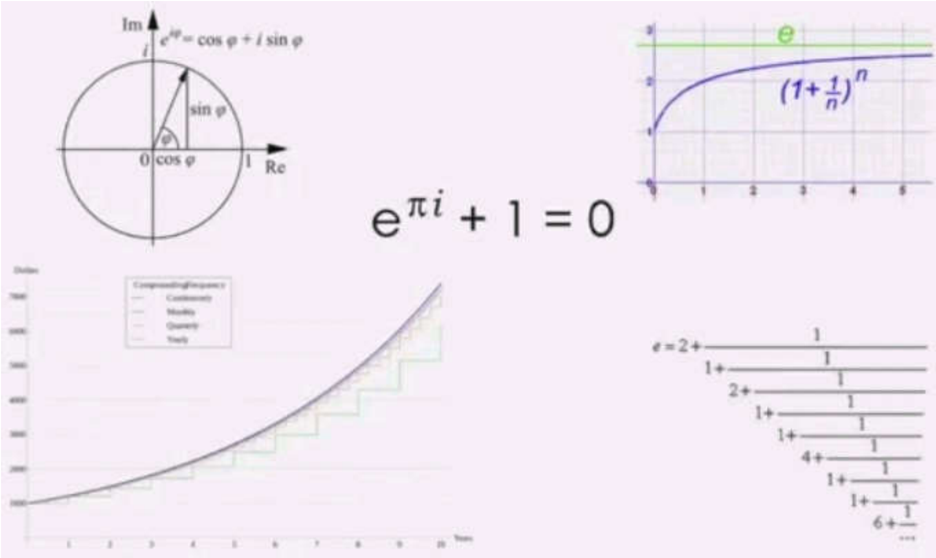
Let us look a little more closely now at Euler's constant. Euler's constant is:

$$e = 2.718{,}281{,}828{,}459{,}045{,}235{,}360{,}287{,}4...$$

Equation 1 : Euler's Constant

Euler's constant is an infinite, no repeating decimal number, which we call a transcendental number in mathematics. You are already familiar with the transcendental number π, 3.141,592,..., that is a number you have probably seen all ready in your studies.

Euler's constant can be thought of as coming from a wide variety of areas, including things like geometry, the interest, when you are talking about interest on a loan, or interest you earn in a bank account, and other situations such as continued fractions. Euler's constant can be derived from a number of these situations, and you will encounter these more later in your mathematical studies.



Let us move on and talk about the function that result from Euler's constant, or the exponential function. Using that notation that Euler developed for functions is called the exponential function. This means we take that Euler constant number, 2.71 etc., and raise it to any number x as the power. For example, $f(1)$ would just be e^1, which is just e itself, or that 2.71 etc.

$$f(x) = e^x$$

The related function to the exponential function is a logarithm function. Here you see our logarithm function:

$$g(x) = \log_e x = \ln x$$

We also denote this as 'ln'. The ln stands for natural logarithm, or it is a special logarithmic function, where the base is e. The base is that little number written as a subscript of the logarithm. We read this as log base e of x, and

$$y = \log_e x \quad \text{means} \quad e^y = x$$

These two functions are very closely related to each other, namely, they are inverse functions for each other. Inverse functions have the property that, when we compose them together, they undo each other, and we are just left with x.

$$f(x) = e^x$$
$$g(x) = \log_e x = \ln x$$
Inverse property :
$$f(g(x)) = e^{\ln x} = x$$
$$g(f(x)) = \ln(e^x) = x$$

That is the inverse property of functions. You may have just seen, we did this with the base of e. E is not the only base you can use when dealing the exponential and logarithm functions. Let us look now at the general exponential and logs.

The general exponential function would just be:

$$f(x) = a^x$$

a is taking the place of some constant. Instead a could be any +number ≠ 1, therefore, is got to be > 0, but we wont let it be = 1. The reason is if a = 1, 1 to any power is just 1, and we get a constant function, which really does not satisfy the properties of the exponential functions that we will be talking about shortly.

Another function we can look at:

$$g(x) = \log_a x$$

Again, that subscript on the log stands for the base. We can convert back and forth between the logarithm and the exponential function according to:

$$f(g(x)) = a^{\log_a x} = x$$
$$g(f(x)) = \log_a(a^x) = x$$

Once again, these two functions have the inverse property. If I compose them together, the exponential and logarithm undo each other, and I am just left with the *x*.

Let us do a quick example so you can see how the log works, because with different bases, sometimes this can be new to students. For example, what if I want to compute:

$$\log_2 16 = ?$$
$$2^4 = 2*2*2*2 = 16$$
$$\log_2 16 = 4$$

You can see logarithm is kind of asking the inverse question of what power do I need to raise the base to get the quantity that I am taking the logarithm of. Let us review some of the exponential properties that you will see later in this course.

You see a lot of properties listed here. You will be talking about these in much more detail later on.

1.1 Exponential Properties

For *a* and *b* +, and *x* and *y* real:

$$a^x a^y = a^{x+y}$$
$$(a^x)^y = a^{xy}$$
$$(ab)^x = a^x b^x$$
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$
$$\frac{a^x}{a^y} = a^{x-y}$$

However, for right now, what you will want to be noticing is that there are a lot of properties of exponential functions, and utilizing these will allow you to solve equations involving exponents, specifically exponential equations with e, or any other base that we are talking about. *a* and *b* here are standing for bases. Remember, those numbers have to be +, and cannot = 1.

The logarithm also has a lot of properties.

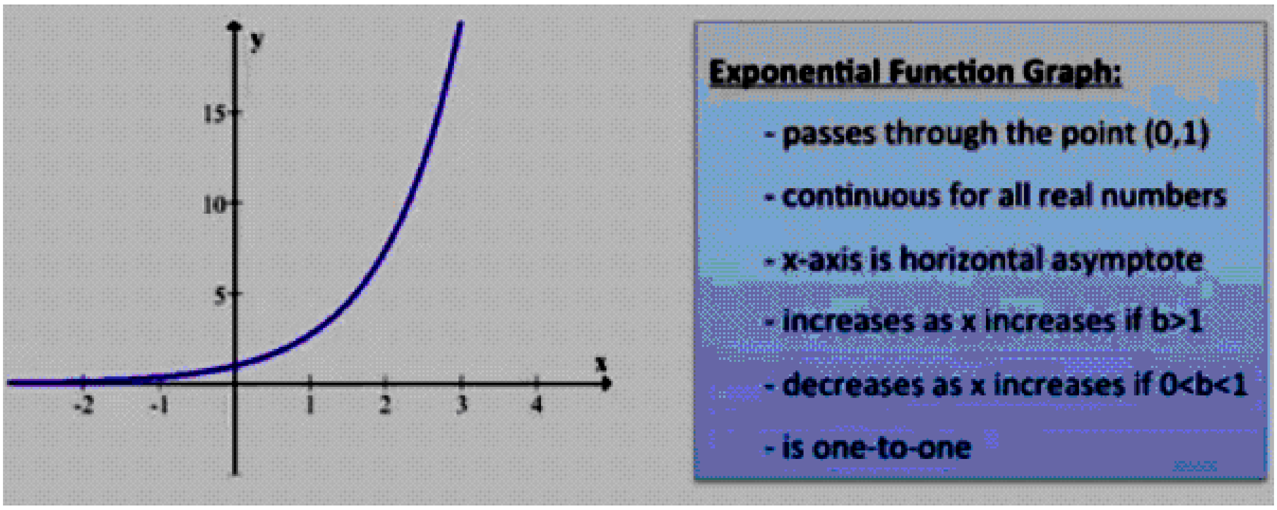
1.2 Logarithm Properties

For *b*, *M* and *N* + real numbers and *p* and *x* real numbers:

$$\log_b 1 = 0$$
$$\log_b b = 1$$
$$\log_b b^x = x$$
$$b\log^{b^x} = x \text{ with } x > 0$$
$$\log_b MN = \log_b M + \log_b N$$
$$\log b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$
$$\log_b M^p = p\log_b M$$

Do not worry about memorizing these now, but you will be utilizing them later to solve logarithm equations. Some of these properties come up again and again in calculus. Therefore, I highly recommend you pay attention now. Learn this well, because it will help you a lot later in your calculus studies.

We can also consider the properties of the graphs of exponentials and logarithms.



Here I have shown you an example of an exponential graph. This is if the value of *a* or the base of the exponential is > 1. If the values were > 0≤ 1, the function would just be going down to the right, and increasing on the left. That would be a -, a base that is smaller, and it would be -growth or decreasing function.

Notice here, I have listed some properties of the exponential function graph. One of the key properties is all exponential functions go through this special point 0, 1. This is, because anything raised to the 0^{th} power just gives you a 1, and that is why that special point is on all of our exponential functions.

Another property you will notice from the graph is this function is continuous. There are no gaps, breaks, or jumps in our curve.

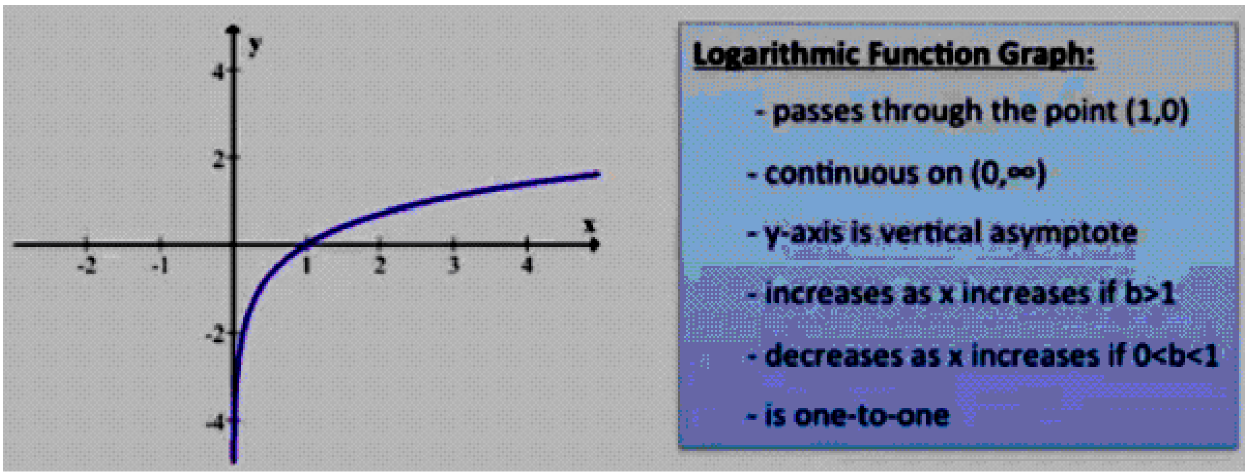
This also has a horizontal asymptote at the x-axis, or $y = 0$. Notice, as I go to smaller values of x, the function gets closer and closer to the x-axis, but it actually never reaches it, and that is why it is an asymptote.

We also have the properties that this function increases as I go to the right, and decreases as I go to the left.

Finally notice this function is 1 : 1. Remember, we checked for functions being 1 : 1 by looking at the horizontal line test. Any horizontal line across my curve will intersect the graph at exactly one point, and not more than one point.

Alright let us now look at the exponential functions inverse, namely the logarithm function for contrast.

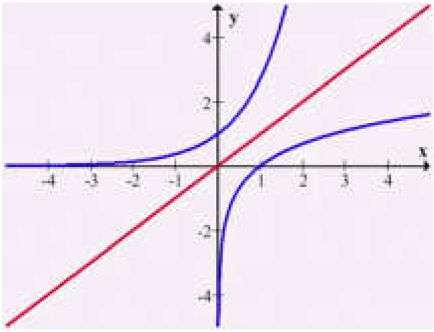
The logarithm function here is given by the graph like this.



A logarithm function graph is also continuous, because there are no breaks, gaps, or jumps. It goes through the special point 1, 0. It also has a vertical asymptote at $x = 0$, or the y- axis. Notice that this function also increases as we go to the right and as I head towards $x = 0$, this function goes to -∞. This function is also 1 : 1, because any horizontal line through this function will intersect the graph, at most, one time.

How are these two functions related to each other?

Well, if you start to look at their graphs, you can probably see that these two functions are inverses, which we have been talking about a lot. If we look at the graphs, you can see that more easily. Here I have graphed a sample of an exponential function with base *e*, and a logarithm function, or natural logarithm, ln(x).



The red line is the line $y = x$. Notice if I folded the screen over the red line, the two curves would line up over each other. That is because they are inverse functions, therefore, they are basically a reflection of across the line $y = x$.

Well, let us see what we are going to learn about exponentials and logarithms in this course. In this unit you will learn to, first of all, evaluate exponential and logarithm expressions. You will also learn to convert back and forth between the exponential and logarithm form of an equation. You will learn to graph those exponential and logarithm functions, particularly when you have transformations. You will learn to solve exponential and logarithm equations. That is going to be one of the most important skills that you will need to take with you to your calculus course.

Finally, we will talk about using exponentials and logarithms to solve application problems. Speaking of applications, what are the applications of exponentials and logarithms? There is quite a few applications including, population dynamics is probably the most standard one you will hear about. If we look at the population, for example the bunnies we looked at earlier growing with time, you will often see exponential growth.

Radioactive decay is another popular application of exponentials and logarithms. You may have heard of carbon dating. Carbon dating is where they figure out the age of old artifacts using the decay of ^{14}C over time, and that is looking at an example of decay; in this case, it is not radioactive. However, we are looking at the decay of a substance. In addition, if you are ever investing money in the bank, you are often getting compound interest. Compound interest is an example of an application of exponential functions.

Newton's law of cooling is the law that tells you, if you put a cake on the counter, how quickly that cake will reach room temperature, or if you put hot soda in the fridge, how quickly that soda will cool. That is Newton's law of cooling, and that is another application of exponential and logarithms.

The Richter scale for earthquakes in California, you may be very familiar with this, is actually a logarithm function, and those numbers that tell you how severe an earthquake is, is actually based on a logarithmic scale.

Sound intensity is another thing that is measured in logarithms. The sound intensity, decibel level, is a logarithmic function.

Finally, the learning curve, which is the rate at which you acquire knowledge, can be modeled closely by exponential and logarithm functions. Therefore, if you are looking to figure out how much information you retain as a function of time at which you are studying, which actually will be modeled by exponentials and logarithms.

2 Solving Exponential Equations

For example, let us solve this equation for x.

$$9^{x+4} = 27^{1-x}$$

Although these bases of this equations are different, they both are powers of 3. Therefore, let us write:

$$\begin{aligned} 9^{x+4} &= 27^{1-x} \\ \left(3^2\right)^{x+4} &= \left(3^3\right)^{1-x} \end{aligned}$$

Now, by properties of exponents, we can multiply the exponent.

$$\begin{aligned} 9^{x+4} &= 27^{1-x} \\ \left(3^2\right)^{x+4} &= \left(3^3\right)^{1-x} \\ 3^{2(x+4)} &= 3^{3(1-x)} \end{aligned}$$

Now these bases are the same. We know that these exponents have to be the same as well, and this is a direct result from the fact exponential functions are 1 : 1. Therefore, we have:

$$\begin{aligned} 9^{x+4} &= 27^{1-x} \\ \left(3^2\right)^{x+4} &= \left(3^3\right)^{1-x} \\ 3^{2(x+4)} &= 3^{3(1-x)} \\ 2(x+4) &= 3(1-x) \\ 2x+8 &= 3-3x \\ 5x &= -5 \\ &\underline{x=-1} \end{aligned}$$

All right, let us look at another example. Let us solve this equation for y:

$$4^{5y-y^2} = 16^{-3}$$

Now the first thing to notice here is that 16 is a power of 4, in fact it is 4^2, is it not? This means we can rewrite this equation as follows.

$$\begin{aligned} 4^{5y-y^2} &= 16^{-3} \\ 4^{5y-y^2} &= \left(4^2\right)^{-3} \end{aligned}$$

Now, on the right we can multiply that exponent which gives us:

$$\begin{aligned} 4^{5y-y^2} &= 16^{-3} \\ 4^{5y-y^2} &= \left(4^2\right)^{-3} \\ 4^{5y-y^2} &= 4^{-6} \end{aligned}$$

Again, now that these bases are the same, these exponents will be as well.

$$\begin{aligned} 4^{5y-y^2} &= 16^{-3} \\ 4^{5y-y^2} &= \left(4^2\right)^{-3} \\ 4^{5y-y^2} &= 4^{-6} \\ 5y-y^2 &= -6 \\ y^2-5y-6 &= 0 \\ (y-6)(y+1) &= 0 \\ &\underline{y=6 \quad or \quad y=-1} \end{aligned}$$

These are a few examples on how we can solve exponential equations. We get the bases the same, and then we equate the exponents.

3 Change of Base Formula for Logarithms

How would we compute:

$$\log_3 5$$

Remember that:

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\text{For example :}$$

$$\log_3 9 = 2$$

$$\text{since} : 3^2 = 9$$

However, looking at our problem over here, do we know a power that we can raise 3 to get 5? We do not, and therefore, we would like to use our calculator, but how? Most calculators, only have two log buttons.

$$\log \to \log_{10}$$
$$\ln \to \log_e$$

However, they do not have a $\log_3$ button. How are we going to compute $\log_3 5$? Well, we can use a change of base formula for log.

Change of Base Formula for Logarithms

For any positive numbers a, b, c such that $(a, c \neq 1)$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Which states, that we can change our base from a to c, as long as we divide by the log of our old base. Therefore, let us apply that here, with $a = 3$, $b = 5$, and $c = 10$, because we have that button on our calculator. We could have also used $c = e$, because we have that button as well. However, let use base 10. That is:

$$\begin{aligned}\log_3 5 &= \frac{\log_{10} 5}{\log_{10} 3} \\ &= \frac{\log 5}{\log 3} \\ &= \underline{1.464,973,520,72\ldots}\end{aligned}$$

Now if we would have used $c = e$, we would have arrived at the same answer. Therefore, this change of base formula for logs is very handy when we need to use the calculator to compute a logarithm.

4 Converting Between Logarithmic and Exponential Equations

For example, let us rewrite this logarithmic equation as an exponential equation.

$$\log_4 \frac{1}{64} = -3$$

We have the following equivalence between logarithmic and exponential forms.

For any numbers a, x, and y, with $a, x > 0$ $(a \neq 1)$

$$\log_a x = y \quad \text{if and only if} \quad a^y = x$$

Notice here that the answer to the logarithm, namely y, is the exponent that we have raised this base 2 to get what we are taking the log of. Therefore, matching our equation to the left-hand side here, we have that the base $a = 4$, $x = {}^{1}/_{64}$ and $y = -3$, and therefore, the exponential form would be:

$$\log_4 \frac{1}{64} = -3$$
$$4^{-3} = \frac{1}{64}$$

This would be our answer.

What about going in the other direction? Let us rewrite this exponential equation as a logarithmic one.

$$3^4 = 81$$

Again, we have this equivalence here, but we will be starting on the right side now. We will have $a = 3$, $y = 4$ and $x = 81$. Now, writing this in the equivalent logarithmic form gives us:

$$3^4 = 81$$
$$\log_3 81 = 4$$

Therefore, we use this equivalence to convert between logarithmic and exponential forms.

5 Properties of Logarithms

For example, given that:

$$\begin{aligned} \log x &= 3 \\ \log y &= -2 \\ \log\left(x^5 y^3\right) &= ? \\ \log \frac{x^2}{\sqrt{y}} &= ? \end{aligned}$$

We will be using the following properties of logs to help us.

Logarithm of a product:	$\log_a(MN) = \log_a M + \log_a N$
Logarithm of a quotient:	$\log_a \frac{M}{N} = \log_a M - \log_a N$
Logarithm of a power:	$\log_a M^p = p \log_a M$

Where *a* is any + base ≠ 1, *M* and *N* are +numbers, and *P* is any number.

Let us begin. By this property, the logarithm of a product is the sum of the logs.

$$\begin{aligned} \log x &= 3 \\ \log y &= -2 \\ \log\left(x^5 y^3\right) &= \log\left(x^5\right) + \log\left(y^3\right) \\ &= 5\log x + 3\log y \\ &= 5(3) + 3(-2) \\ &= 15 - 6 \underline{= 9} \end{aligned}$$

By this last property, the logarithm of a power, we can take this power and bring it down in front of the logs.

$$\begin{aligned} \log x &= 3 \\ \log y &= -2 \\ \log \frac{x^2}{\sqrt{y}} &= \log x^2 - \log \sqrt{y} \\ &= \log x^2 - \log y^{1/2} \\ &= 2\log x + \frac{1}{2}\log y \\ &= 6 - \left(\frac{1}{2} - 2\right) \\ &= 6 + 1 \underline{= 7} \end{aligned}$$

By the second property over here, the logarithm of the quotient is equal to the difference in the logarithms. This is log $\sqrt{y}$ = log $y^{1/2}$, and that is ½ * log y!

Alright, let us look at another example. Let us compute:

$$\log_2 24 - \log_2 3$$

Now do we know a power that we can raise 2 to get 24 or a power we can raise 2 to get 3? We do not, therefore, we could use a change of base formula on each of these logs separately, but it is going to much easier if we use these properties of logs. If we look here at this middle property, we can use it in the reverse direction. In other words:

$$\begin{aligned} \log_2 24 - \log_2 3 &= \log_2\left(\frac{24}{3}\right) \\ &= \log_2(8) \\ &\underline{= 3} \\ \text{since}: 2^3 &= 8 \end{aligned}$$

6 Evaluating Logarithm Expressions

For example, let us evaluate:

$$\log_2 32$$

Now, we have the following definition of the logarithm:

$$\begin{array}{lll} \log_a x = y & \text{if \& only if} & a^y = x \\ & \text{with:} & \\ \text{a \& x are positive} & \text{and} & a \neq 1 \end{array}$$

That is the answer why to a logarithm is the exponent that we need to raise the base to, to get what we are taking the logarithm of. Let us apply this here.

$$\begin{aligned} y = \log_2 32 &\Leftrightarrow 2^y = 32 \\ \underline{y = 5} & \\ \text{since:} & \\ 2^5 &= 2 * 2 * 2 * 2 * 2 \\ &\underline{= 32} \end{aligned}$$

Let us look at another example. Let us evaluate:

$$\begin{aligned} &\log_{16} \frac{1}{4} \\ y &= \log_{16} \frac{1}{4} \Leftrightarrow 16^y = \frac{1}{4} \\ \left(4^2\right)^y &= 4^{-1} \\ 4^{2y} &= 4^{-1} \\ 2y &= -1 \\ &\underline{y = -\frac{1}{2}} \end{aligned}$$

Okay, let us look at one more. Let us evaluate:

$$\begin{aligned} &\log_{27} 9 \\ y &= \log_{27} 9 \Leftrightarrow 27^y = 9 \\ \left(3^3\right)^y &= \left(3^2\right) \\ 3^{3y} &= 3^2 \\ 3y &= 2 \\ &\underline{y = \frac{2}{3}} \end{aligned}$$

7 Solving Logarithmic Equations

For example, let us solve this equation for *x*.

$$8 + \log_5 (x + 4) = 9$$

We can begin by bringing in the 8 to the right-hand side, which would give us that:

$$\log_5 (x + 4) = 1$$

And now writing this in exponential form gives us:

$$\begin{aligned} \log_5 (x + 4) &= 1 \\ 5^1 &= x + 4 \\ &\underline{x = 1} \end{aligned}$$

Remember that:

$$\log_a x = y$$
$$a^y = x$$

Now, in solving logarithmic equations it is very important for us to check our answers. Let us do that.

$$\begin{aligned} 8+\log_5(x+4) &= 9 \\ 8+\log_5(1+4) &= 9 \\ 8+\log_5(5) &= 9 \\ 8+1 &= 9 \\ \underline{9 = 9} \end{aligned}$$

It works. Therefore, our answer is $x = 1$.

Alright, let us see another example. Let us solve this equation for x.

$$\log(x-15)+\log x = 2$$

We can begin here by condensing the left-hand side into a single logarithm by using the following property.

$$\log A + \log B = \log(AB)$$

Then we can convert this into exponential form again by using the equivalence, and remembering though that log means $\log_{10}$. That is, this equation is equivalent to:

$$\begin{aligned} \log(x-15)+\log x = 2 &\rightarrow \log[(x\text{-}15)x] = 2 \\ 10^2 &= (x\text{-}15)x \\ 100 &= x^2\text{-}15x \\ x^2\text{-}15x\text{-}100 &= 0 \\ (x\text{-}20)(x+5) &= 0 \\ \underline{x = 20 \; or \quad \cancel{x=-5}} \end{aligned}$$

However, remember we have to check these answers. Therefore, let us start with $x = 20$. We will plug it in:

$$\begin{aligned} \log(x-15)+\log x &= 2 \\ \log(20-15)+\log(20) &= 2 \\ \log(5)+\log(20) &= 2 \\ \log(100) &= 2 \\ \underline{10^2 = 100} \end{aligned}$$

x = 20 works. It is a solution. What about $x = -5$? Well, that would require us to plug in a -5 in our equation.

$$\log(-5-15)+\log(-5) = 2$$

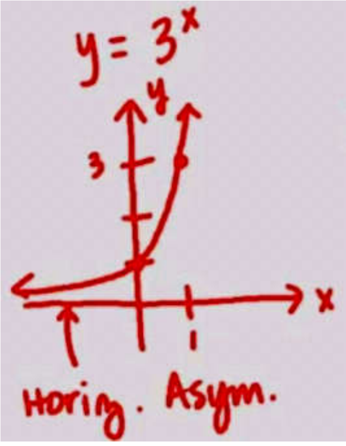
Because log(-5-15), which is log(-20), and log(-5) are both undefined, because remember that the domain of the logarithm is values are strictly > 0. Which means, we are going to cross off $x = -5$ over here, and therefore, our only answer is x = 20. It is very important to check your answers when solving logarithmic equations.

8 Exponential Graphs

For example, let us sketch the graph of:

$$f(x) = 3^{x-1} - 2$$

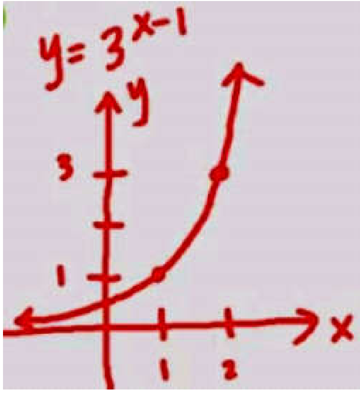
Then we are going to find any x or y-intercepts of its graph. Let us use graph transformations to help us here. Let us start by sketching $y = 3^x$. What does it look like?



Well, it has a y-intercept at 1, and then, when $x = 1$, $y = 3$. Therefore, we have the point 1, 3, which lies on the graph. The exponential function looks like this. Now, the x-axis or $y = 0$ is a horizontal asymptote.

$$f(x) = 3^{x-1} - 2$$

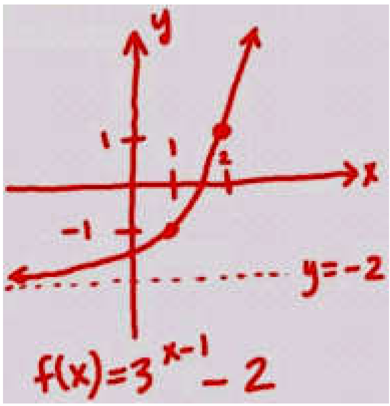
Now, what does this -1 do here? What that does is it shifts this graph rigidly one unit to the right. That is the graph of $y = 3^{x-1}$ looks like this.



We shift our graph one unit to the right? This is going to move to 0 + (1, 1) or (1, 1). What is going to happen to the point 1, 3? It is going to move to 1 + (1, 3) or (2, 3). Moreover, the horizontal asymptote will still remain at y = 0.

$$f(x) = 3^{x-1} - 2$$

Now, what does this -2 do to this graph? What that does is it shifts this entire graph rigidly down two units.



The horizontal asymptote that was at $y = 0$ is also shifting down two units, which means that the new horizontal asymptote then will be at $y = -2$. Therefore, our graph looks like this.

Now we are also asked to find in the x- or y-intercepts of its graph. Looking at the graph, we see that we have an y-intercept and an x-intercept. Let us find these intercepts.

To find the y-intercept, we set $x = 0$ in our equation. To find our x-intercept, we set $y = 0$.

8.1 Intercepts

$$f(x) = 3^{x-1} - 2$$

$$\begin{aligned} &\text{y}-\text{intercept:}\\ x &= 0\\ y &= 3^{0\text{-}1}\text{-}2\\ y &= 3^{\text{-}1}\text{-}2\\ y&\frac{1}{3}-2\\ y &= \text{-}\frac{5}{3} \end{aligned}$$

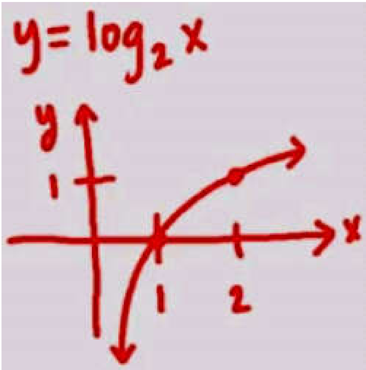
$$\begin{aligned} &\text{x}-\text{intercept:}\\ y &= 0\\ 0 &= 3^{x\text{-}1}\text{-}2\\ 3^{x\text{-}1} &= 2\\ \log_3 2 &= x\text{-}1\\ x &= \log_3 2 + 1 \end{aligned}$$

9 Logarithmic Graphs

For example, let us sketch the graph of:

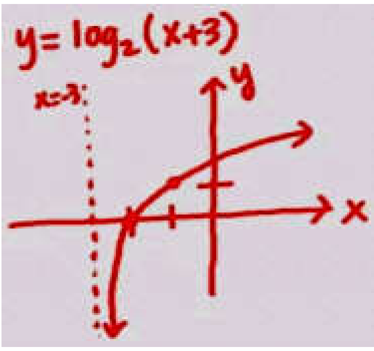
$$f(x) = 1 + \log_2(x+3)$$

Then we will find any x- or y-intercepts of its graph. Let us sketch this by using graph transformations. First of all, what does $y = \log_2 x$ look like? It is looks like this, where the x-intercept is 1.



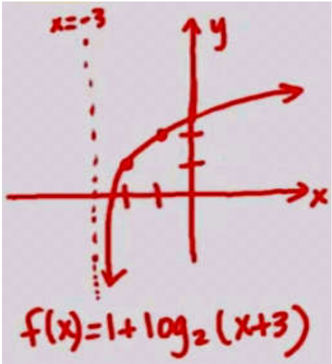
$$f(x) = 1 + \log_2(x+3)$$

Now what does this +3 do here? By adding 3 + x, what that does is, it rigidly shifts these graph three units to the left. Remember, we originally had a vertical asymptote here at $x = 0$. Suppose shifting everything to the left three units, then the asymptote will be shifted to $x = -3$. Therefore, the graph will look like the following.



$$f(x) = 1 + \log_2(x+3)$$

Alright, and finally what does this +1 do here? What that does is, it shifts this graph rigidly up one unit, and our graph then looks like this. Our vertical asymptote is still at x = -3



We are also asked to find any x- or y-intercepts, and looking at our graph, we have an x-intercept and an y-intercept. Let us find out what these values are, let us find these intercepts.

9.1 Intercepts

$$f(x) = 1 + \log_2(x+3)$$

y − intercept :

$$x = 0$$
$$y = 1 + \log_2(0+3)$$
$$\underline{y = 1 + \log_2 3}$$

x − int*ercept:*

$$y = 0$$
$$0 = 1 + \log_2(x+3)$$
$$\log_2(x+3) = -1$$
$$2^{-1} = x + 3$$
$$\frac{1}{2} = x + 3$$
$$x = \frac{1}{2} - 3$$
$$\underline{x = -\frac{5}{2}}$$

Angles and their Measure, Trigonometric Functions: A Unit Circle, Solving Right Triangles

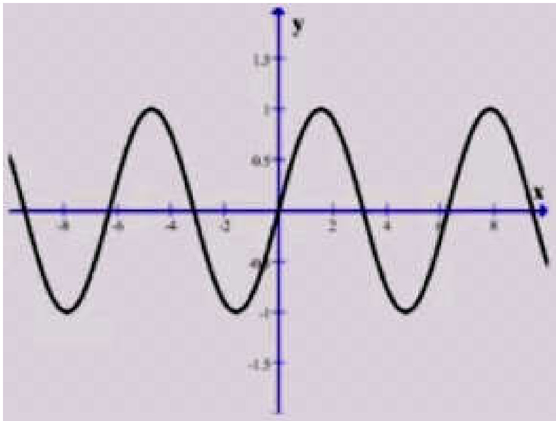
1 Introduction to Trigonometry

In this unit, we will be learning about trigonometry.

Trigonometry is the study of angles of triangles. Let us look at the technical definition. Trigonometry is a branch of mathematics, that deals with the relationships between sides and angles of triangles, and the calculation based on them, particularly involving trigonometric functions.

You can think of trigonometry as kind of having four components. We want to study both, triangles, circles, and the relationships between the angles of these objects. We look at three main trigonometric functions, sine, cosine, tangent, and we want to study oscillatory behavior.

As we see in this graph here a function that oscillates back and forth.



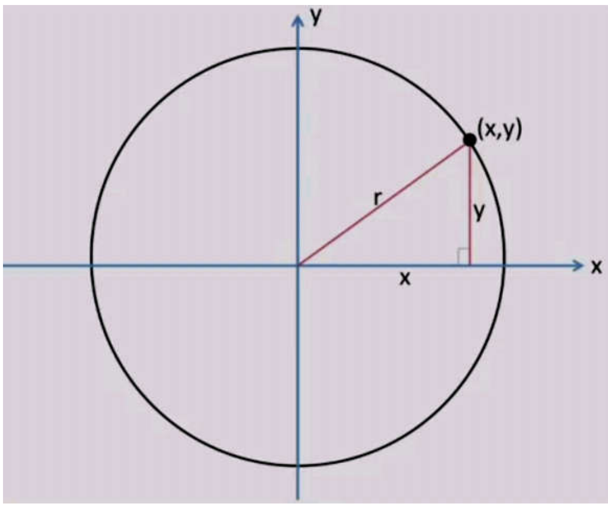
Let us look at a simple example.

When you were a kid, you might have had a slinky toy. Well, the slinky toy is a great example of oscillatory behavior. Notice, if I take my slinky and bounce it, I get some oscillations. The slinky oscillates back and forth; you can also take the slinky and make some waves. When I make waves with my slinky, that is another example of oscillatory behavior.

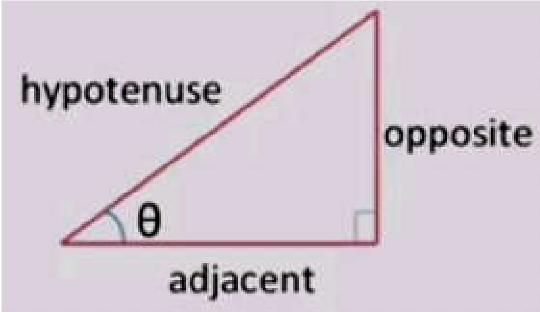
Notice that the polynomials, we have been studying so far, would not be able to account for this type of behavior. Polynomials can have some wiggles to them, but, eventually, they tend either to ∞ or to -∞. Therefore, we need some sort of function with periodic behavior.

By periodic I mean it oscillates with time, and does the same thing over and over again, forever and ever. We are not going to play with the slinky forever though.

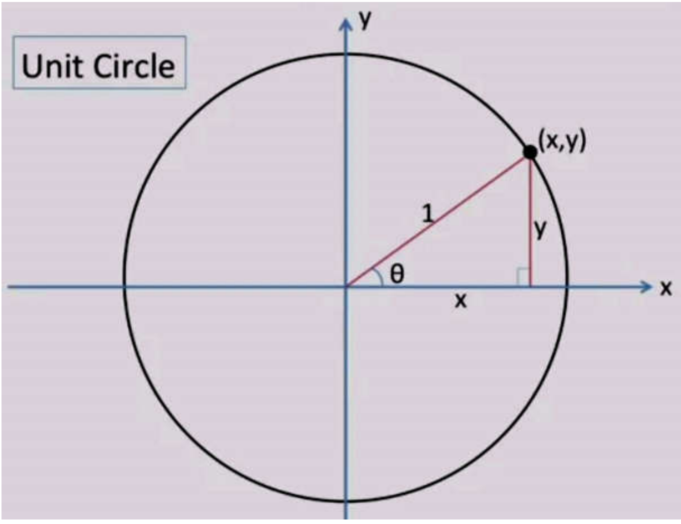
Let us look at the correspondence between circles and triangles. You might not think these two things have a lot in common. If we consider a circle of radius *r*, and we think of a point on this circle, with coordinates (*x*, *y*), we could create a triangle from this circle by dropping a perpendicular from the point (*x*, *y*) onto the x-axis.



Then we have the triangle, labeled as you see here.



When I replace that radius with 1, therefore, take the circle I had before, replace the radius with 1, we get a unit circle.



We call the angle that that radial line makes with the x-axis, an angle θ. θ is just a Greek letter that we often use for angle. There is nothing special about the letter θ, it is just a convenient way to let people know we are talking about an angle there.

If we just look at the triangle from this picture, we can think about, well, how we would determine what θ was, if we knew some of the sides from that right triangle.

Therefore, with my right triangle I have three kind of main sides to the triangle. There is the hypotenuse, the side opposite the right angle. There is the adjacent side, the side next to the angle I am interested in θ, and the opposite side, the side on the opposite side of the triangle from θ.

This brings us to those trigonometric functions we mentioned before. We have three main trigonometric functions we will talk about, that are the sine, the cosine, and the tangent. These three functions can be related to the right triangle, via the relationship that the sine is the opposite side over the hypotenuse. The cosine is the adjacent side over the hypotenuse, and the tangent of an angle is the opposite side over the adjacent side.

Trigonometric Functions

$$\sin\theta = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos\theta = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan\theta = \frac{\textit{opposite}}{\textit{adjacent}}$$

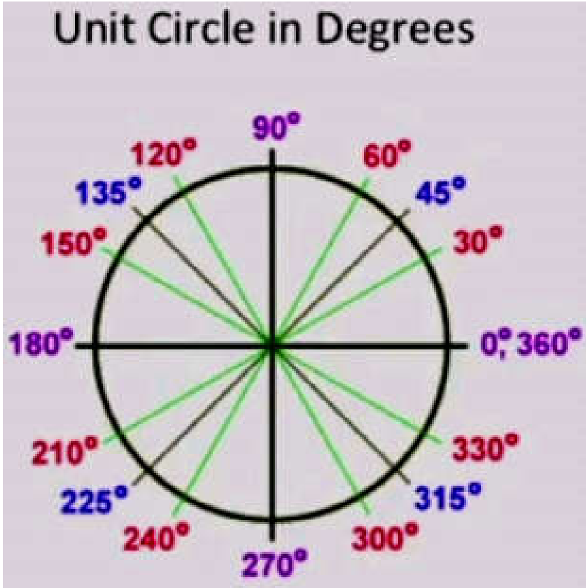
A lot of students like to use the mnemonic, soh cah toa, to help you remember this.

This basically is: soh; sign is opposite over hypotenuse, cah; cosine is adjacent over hypotenuse, and toa; tangent is opposite over adjacent.

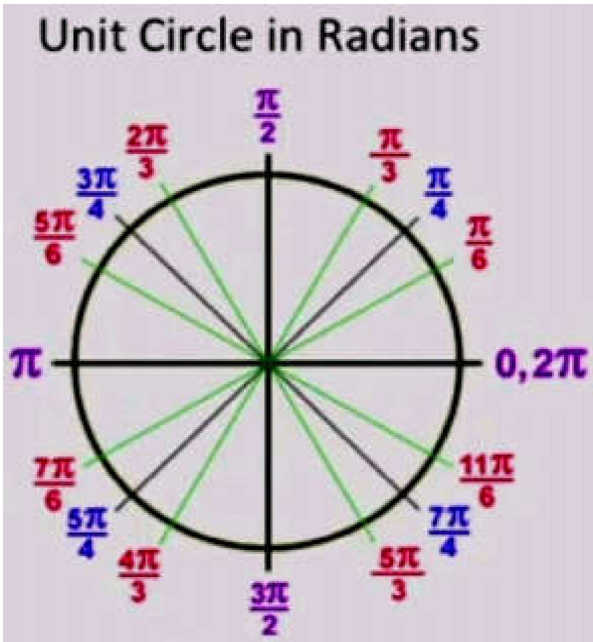
Let us take some more look at these trigonometric functions.

We said that this triangle was also related to the unit circle, or any circle for that matter. The unit circle is just a special circle with r = 1. On that unit circle, there are several angles that we study a lot, because these are values that we can evaluate explicitly for a circle.

Those points (x, y) on the circle, that are interesting or we know the values of, would be things going from the x-axis along the unit circle, to angles like 30°, 45°, 60°, 90°, et cetera. These special angles are just values for which the (x, y)-coordinates on the unit circle are easy to find.



We do not always work with degrees in mathematics when we are looking at trigonometric functions. We also work with something called radians.



Most of you probably know a circle has 360°. What you may not know is that we also can say a circle has 2 * π radians. The relationship between degrees and radians is simply that:

$$360^\circ = 2\pi \text{radians}$$

Therefore, in these two circles I am just depicting the degree value and radian value of some key angles on the unit circle. We can also look at the trigonometric values of the trigonometric functions at those key angles.

For example, I can make a table of my sine, cosine, and tangent, with those angles θ, as the input. Notice here, we have some values. By the end of this course, you definitely want to memorize this table.

1.1 Table of Trigonometric Function Values

Table of Trigonometric Function Values

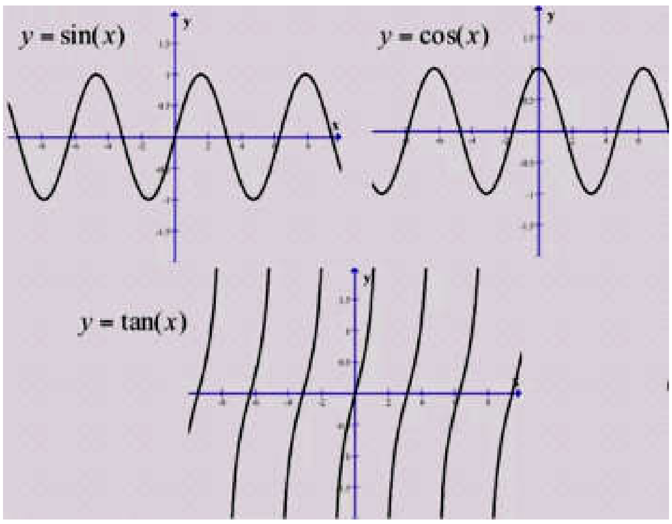
θ	0 0	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	U
$\csc(\theta)$	U	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec(\theta)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	U
$\cot(\theta)$	U	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

It takes a little bit of work, but it will be worth it, particularly when you get to Calculus. Being a master of trigonometric values is super important.

Notice on this table, I snuck in three extra functions. I have not talked much about these three extra trigonometric functions, because they are basically directly related to the ones you already know, sine, cosine, and tangent.

$$\boxed{\begin{aligned}\csc\theta &= \frac{1}{\sin\theta}\\ \sec\theta &= \frac{1}{\cos\theta}\\ \cot\theta &= \frac{\cos\theta}{\sin\theta}\end{aligned}}$$

Let us look at our three main trigonometric functions. I said that they correspond to oscillatory behavior, or an oscillating function. Just looking at a triangle, it may not be clear to you that those are oscillatory functions, but basically, as I change an angle θ and input it into this function, I get differing values, and these values repeat with time.



Each of these functions has what we call a characteristic period, or the amount of time before that function repeats again. Up on this graph you see our sine, our cosine, and our tangent graphs. You will be working with these more in future units.

Another thing that you will be learning a lot about is trigonometric identities. Trigonometric identities are basically relationships between our trigonometric functions, which help us solve equations and manipulate expressions involving the trigonometric functions. There is quite a few of these, many of them you will want to memorize.

1.2 Trigonometric Identities[2]

$$\left.\begin{array}{l}\tan x\ \cot x = 1\\ \cos x\ \sec x = 1\\ \sin x\ \csc x = 1\end{array}\right\}\text{Reciprocal}\qquad \left.\begin{array}{l}\cos^2 x + \sin^2 x = 1\\ 1+\tan^2 x = \sec^2 x\\ 1+\cot^2 x = \csc^2 x\end{array}\right\}\text{Pythagorean}$$

$$\left.\begin{array}{l}\sin(-x) = -\sin x\\ \tan(-x) = -\tan x\\ \cos(-x) = \cos x\end{array}\right\}\text{Odd - Even}\left\{\begin{array}{l}\csc(-x) = -\csc x\\ \cot(-x) = -\cot x\\ \sec(-x) = \sec x\end{array}\right.$$

$$\left.\begin{array}{l}\sin x = \cos\left(\frac{\pi}{2}-x\right)\\ \tan x = \cot\left(\frac{\pi}{2}-x\right)\\ \sec x = \csc\left(\frac{\pi}{2}-x\right)\end{array}\right\}\text{Cofunction}\left\{\begin{array}{l}\cos x = \sin\left(\frac{\pi}{2}-x\right)\\ \cot x = \tan\left(\frac{\pi}{2}-x\right)\\ \csc x = \sec\left(\frac{\pi}{2}-x\right)\end{array}\right.$$

$$\left.\begin{array}{l}\sin 2x = 2\sin x\cos x\\ \cos 2x = \cos^2 x - \sin^2 x\\ \cos 2x = 2\cos^2 x - 1\\ \cos 2x = 1 - 2\sin^2 x\\ \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}\end{array}\right\}\text{Double Angle}\left\{\begin{array}{l}\sin^2 x = \dfrac{1-\cos 2x}{2}\\ \cos^2 x = \dfrac{1+\cos 2x}{2}\\ \tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}\end{array}\right.$$

$$\left.\begin{array}{l}\cos(A+B) = \cos A\cos B - \sin A\sin B\\ \sin(A+B) = \sin A\cos B + \cos A\sin B\\ \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}\end{array}\right\}\text{Sum \& Difference}\left\{\begin{array}{l}\cos(A-B) = \cos A\cos B + \sin A\sin B\\ \sin(A-B) = \sin A\cos B - \cos A\sin B\\ \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}\end{array}\right.$$

$$\left.\begin{array}{l}\tan x = \dfrac{\sin x}{\cos x} = \dfrac{\sec x}{\csc x}\\ \cot x = \dfrac{\cos x}{\sin x} = \dfrac{\csc x}{\sec x}\end{array}\right\}\text{Quotient}$$

$$\left.\begin{array}{l}\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}\\ \sin A - \sin B = 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2}\\ \cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}\\ \cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}\end{array}\right\}\text{Sum to Product}$$

$$\left.\begin{array}{l}\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\\ \cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]\\ \sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\end{array}\right\}\text{Product to Sum}$$

$$\left.\begin{array}{l}\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}\\ \cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}\end{array}\right\}\text{Half Angle}\left\{\begin{array}{l}\tan\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}\\ \tan\dfrac{x}{2} = \dfrac{1-\cos x}{\sin x}\\ \tan\dfrac{x}{2} = \dfrac{\sin x}{1+\cos x}\end{array}\right.$$

Some of them you will want to make sure you know how to derive, or figure out what the relationship between these quantities is. You will be spending a lot of time working with trigonometric identities. One trigonometric identity you may have already seen is the Pythagorean Theorem; you may have seen this in a geometry course. The Pythagorean Theorem says that

$$a^2 + b^2 = c^2$$

If you remember what that means for a triangle, one side squared plus the other side squared gives you that hypotenuse squared. The way we say this with trigonometric functions is:

$$\sin^2\theta + \cos^2\theta = 1$$

This is just an example to give you a flavor for what these trigonometric identities are all about they are relationships between your trigonometric functions, which give you an expression that might be helpful in evaluating some sort of equation.

In this unit, you are going to learn several things about trigonometric functions. First, you are going to learn how to work with angles, convert back and forth between degrees and radians, and figure out how to figure out which angle you are talking about. Do you want the small or acute angle or the larger obtuse angle when you are drawing these pictures? You will also learn how to utilize right triangles and the unit circle, to find angles and lengths in triangles and points on the circle. Finally, you are going to evaluate in graph trigonometric functions, and use those trigonometric identities that we talked about. You will solve trigonometric equations, and then, most importantly, you will learn to use trigonometry to solve real world applications.

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[2] Trigonometric identities

1.3 Applications of Trigonometry

What might those real world applications be?

Well, applications of trigonometry include things like astronomy. Trigonometry was actually first developed to help talk about astronomy problems. They wanted to figure out things like, where bodies were going to be in the sky at different periods of time, and that involves a lot of triangles, if you think about it. If I know the planet was here one day, and up there the next, I am making some sort of angle that I can use trigonometry to deal with.

Other things include geography, optics, architecture, mechanics, seasonal phenomenon, anything that changes with time, and some sort of oscillatory behavior, like the seasons, or the temperature as a function of time throughout the year is something that might be suitable for study with trigonometric functions.

Signal processing is another very popular area to use trigonometry with, because most signals you can think of as waves, and studying those waves involve trigonometric functions.

2 Converting Between Degrees-Minutes-Seconds and Degrees

For example, let us convert 5°13′11″ to degrees.

1° can be divided into 60 parts called minutes, and the notation "′" means minutes. These minutes are further divided into 60 parts called seconds, and the notation "″" means seconds. It should be pointed out that the words *minute* and *second* used in this context have no immediate connection to how these words are usually used as amounts of time.

Therefore, we have the following:

$$\begin{aligned}
1^\circ &= 60' \\
1' &= 60'' \\
60' &= (60)*(60)'' \\
&= 3600'' \\
1^\circ &= 3600'' \\
1' &= \left(\frac{1}{60}\right)^\circ \\
1'' &= \left(\frac{1}{3600}\right)^\circ
\end{aligned}$$

These are the equations that we will be using to help us convert to degrees. Namely, this is:

$$\begin{aligned}
5^\circ 13' 11'' &= 5^\circ + 13' + 11'' \\
&= 5^\circ + 13\left(\frac{1}{60}\right)^\circ + 11\left(\frac{1}{3600}\right)^\circ \\
&= \underline{5.2197\overline{2}^\circ}
\end{aligned}$$

Let us look at going the other way. Let us convert 41.795° to degree-minute-second form.

$$\begin{aligned}
41.795^\circ &= 41^\circ + .795^\circ \\
&= 41^\circ + .795(60)' \\
&= 41^\circ + 47.7' \\
&= 41^\circ + 47' + .7' \\
&= 41^\circ + 47' + .7(60)'' \\
&= \underline{41^\circ 47' 42''}
\end{aligned}$$

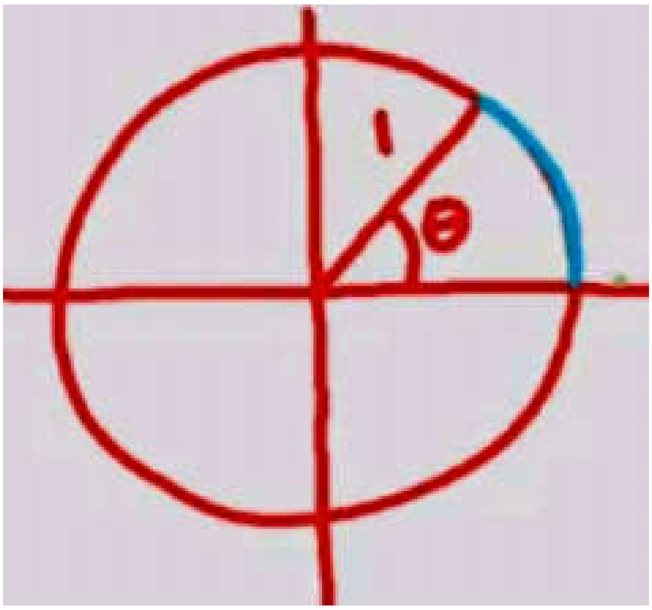
3 Converting Between Degrees and Radians

For example, let us convert 120° to radian measure in terms of π.

Now we have the following conversion formulas.

$$\begin{aligned}
1^\circ &= \frac{\pi}{180}\,\textit{radians} \\
1\textit{radian} &= \left(\frac{180}{\pi}\right)^\circ
\end{aligned}$$

Before applying these though, let us take a minute to think about why they are true. Let us look at the unit circle, which is a circle with r = 1. Remember that the radian measure of this angle here is the length of this corresponding arc.



What would the radian measure be of one full rotation? Would that be not the length all the way around this unit circle? That is the circumference of the circle, which is 2 * π * r, which is 1. Therefore, one rotation measures 2π radians.

What is the degree measure of one full rotation is not that 360°? Therefore, regardless of what measure we use to measure one full rotation, it has to be the same, that is:

$$2\pi \, radians = 360^\circ$$
$$\pi \, radians = 180^\circ$$

We will use this first equation when we convert from degrees to radians, and we will use the second equation when we convert from radians to degrees.

Here we are asked to convert from degrees to radians. We will be using this first conversion.

$$120^\circ = 120\left(\frac{\pi}{180}\right) radians \quad | \div 60$$
$$= \underline{\frac{2\pi}{3}}$$

Sometimes, when we measure in radians, the word radians will be dropped!

Alright, let us see another example. Let us convert:

$$-\frac{5\,\pi}{4} \; radians \to {}^\circ$$

Again, we have the following conversions, but since we are going from radians to degrees, we are going to be using the second one now.

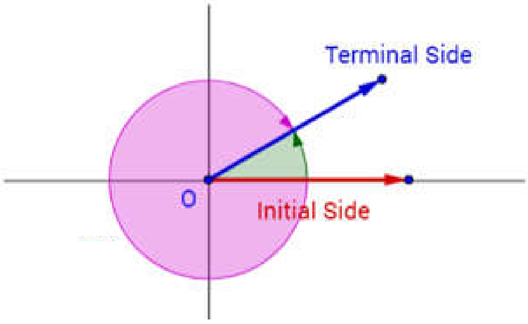
$$1^\circ = \frac{\pi}{180} radians$$
$$1 radian = \left(\frac{180}{\pi}\right)^\circ$$

That is:

$$-\frac{5\,\pi}{4} \; radians = -\frac{5\,\pi}{4}\left(\frac{180}{\pi}\right)^\circ \qquad | \; shorten$$
$$= -5 * 45^\circ$$
$$= \underline{-225^\circ}$$

4 Co-terminal Angles

For example, let us find an angle $0 < \theta > 2\pi$ that is co-terminal with $\theta = -\dfrac{11\pi}{6}$



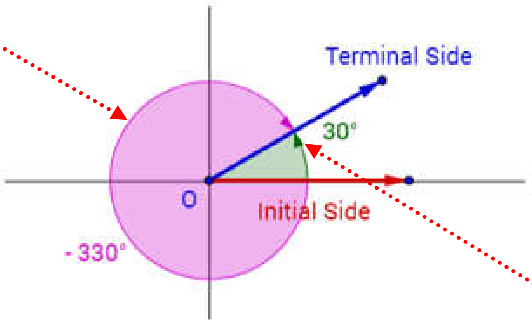
Now, two angles are co-terminal, if they share the same initial and terminal sides. Let us sketch our angle here in standard position. Now, an angle is in standard position when its initial side is on the +x-axis. Because our angle here is -, that means that we are going to start rotating in the clockwise direction.

How far are we going to go? Well, one complete revolution in a clockwise direction measures -2π. How many sixths is -2π?

$$-2\pi = -\frac{12\pi}{6}$$

Which means that our angle here, $-\frac{11\pi}{6}$ did not quite make it there, because 11 < 12, which means we are $\frac{\pi}{6}$ short of -2π.

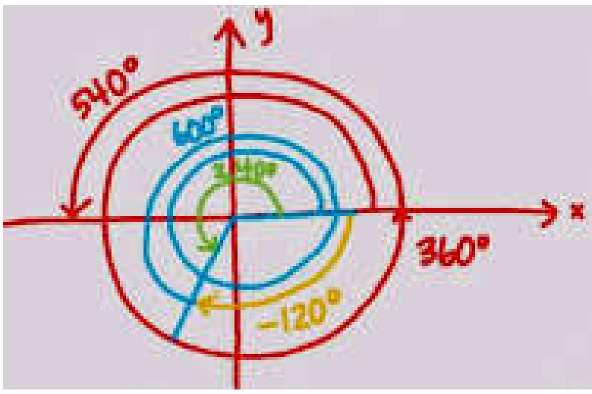
Therefore, here is our angle $-\frac{11\pi}{6}$.



The angle $0 < \theta > 2\pi$ that we are looking for has the same initial and terminal side as this angle here, $\frac{\pi}{6}$, which would be our answer. All right, let us look at another one.

$$\theta = 600°$$

Find an angle $0° < \theta > 360°$ that is co-terminal with θ. Again, let us sketch θ in standard position.



Its initial side will be the +x-axis, and, because the angle is +, we are going to start rotating in the counter-clockwise direction. How far do we go? Well, one complete revolution measures 360°. Therefore, if we go another 180°, where we are? Well, 360° + 180° = 540°; this angle here would be equal to 540°.

We want to get to 600°, which means we are 60° short. We need to go another 60° in a counter-clockwise direction to get to 600°. Namely we go one full revolution, and we go another half, and we go 60 more. Therefore, this would be 600° here.

All right, we are first looking for an angle $0^\circ < \theta > 360^\circ$ that is co-terminal with this angle. Would that just not be the angle 180° + 60°, or 240°? Both of these angles have the same initial and terminal sides, and, moreover, 240° is between 0 and 360°. Therefore, 240° would be our answer.

Now we still need to find an angle $0^\circ < \theta > -360^\circ$ that is co--terminal with θ.

With negative measured angles, we go counter-clockwise from the initial side. Would that not be the angle of -120°, because a quarter turn in the clockwise direction measures -90°, and then we have to go 30° more. Therefore, that would be -120°, and our answers would be:

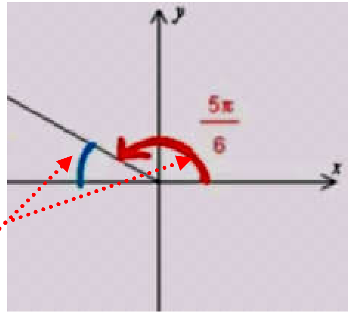
240°

-120°

5 Reference Angles

For example, let us find the reference angle for $\theta = \frac{5\pi}{6}$.

Now, the reference angle is the acute positive angle formed by the terminal side of θ and the x-axis.



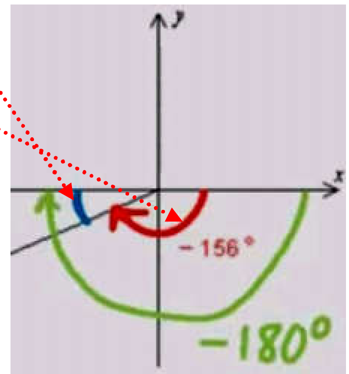
$\frac{5\pi}{6}$ is here and then, this here is the reference angle. It is the acute angle formed by the terminal side of θ and the x-axis. So, what is this acute angle? Well, this angle here measures π radians, and we can think:

$$\pi = \frac{6\pi}{6}$$

Our reference angle is = $\pi/6$.

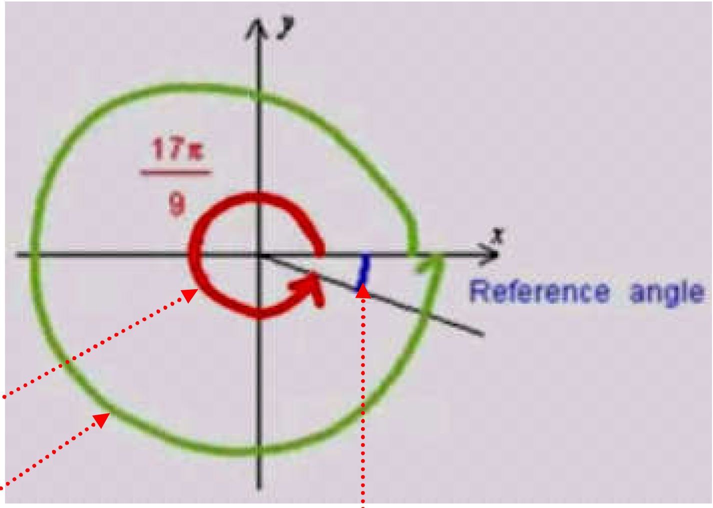
Alright, let us look at another example. Let us find the reference angle for θ = -156°.

Well, here is -156° and this angle here is the reference angle or the acute angle formed from the terminal side of the angle in the x-axis.



Now, the complete angle here measures -180°. -156° did not quite reach -180°, therefore, how much was it short? Well, since 180° - 156° = 24°, then this reference angle is this positive acute measure here, which would be our answer.

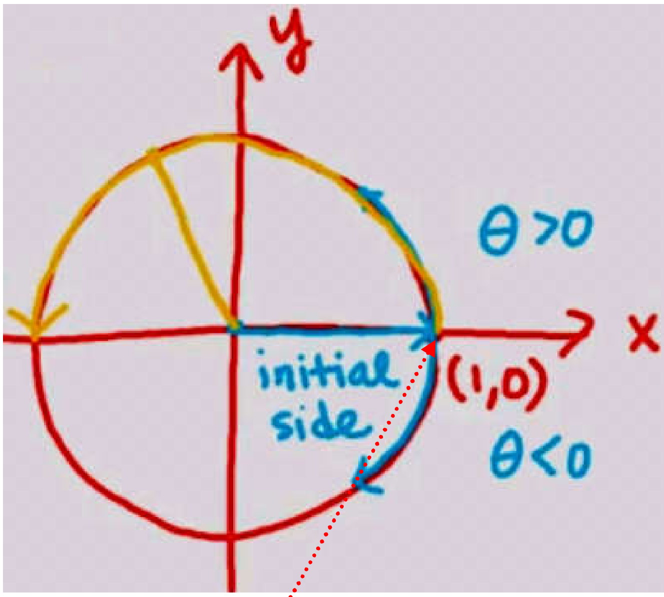
Alright, let us look at one more. Let us find the reference angle for $\theta = \frac{17\pi}{9}$.



Here is $\frac{17\pi}{9}$, that is almost one complete revolution. Remember, the radian measure of one complete revolution is 2π radians, or thinking of that in terms of ninths, it is $\frac{18\pi}{9}$. Therefore, $\frac{17\pi}{9}$ is $^1/_9$ short of one complete revolution. That is, this reference angle here we are looking for them is $\frac{\pi}{9}$. Therefore, the reference angle then is simply the acute angle formed by the terminal side of θ and the x-axis.

6 Sketching an Angle in Standard Position

For example, let us sketch in standard position. Let us begin by drawing a unit circle here.



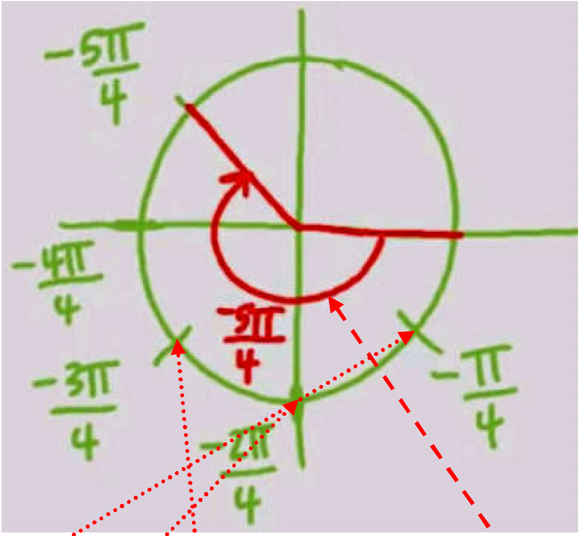
A unit circle is a circle of r = 1, which means this is the point (1, 0). Now, an angle is in standard position if its initial side is on the +x-axis. If $\theta > 0$, we rotate in a counter-clockwise direction, and if $\theta < 0$, then we rotate in a clockwise direction. Since our angle $\theta = \frac{2\pi}{3}$ is > 0, we are going to be rotating in the counter-clockwise direction. However, where do we stop?

Well, one complete rotation measures 2π radians, because, remember, the radian measure of an angle is defined as the length of the corresponding arc on the unit circle. Since the circumference of the unit circle is 2π, then the radian measure of one complete revolution is 2π radians.

Moreover, ½ of the rotation would measure π radians, which means, if we started rotating our initial side all the way to here, this would be π radians or $^{3\pi}/_3$. That means our angle here did not quite get there. We are $^{\pi}/_3$ short of it.

Alright, let us take a look at another example. Let us sketch the angle $\theta = -\frac{5\pi}{4}$ in standard position.

Again, we will begin by drawing a unit circle.



Here is the initial side on the +x-axis, and because our angle up here is < 0, we are going to be rotating in the clockwise direction. Again, where do we stop? Remember that ½ of a rotation measures $-\pi$ radians in the clockwise direction. Moreover, how many fourths is $-\pi$? Is that not $-^{4\pi}/_{4}$?

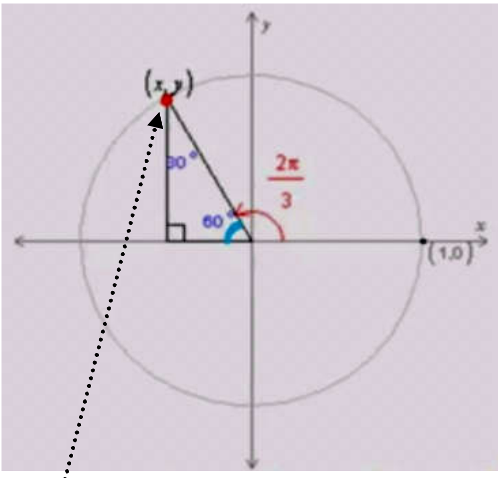
Therefore, this is $1 * -^{\pi}/_{4}$, this is $2 * -^{\pi}/_{4}$, this is $3 * -^{\pi}/_{4}$, this a.s.o, which is our angle. Sketching it in standard position, this is the initial side and this is the terminal side; our angle is here.

7 Trigonometric Values of Special Angles

For example, let us find the exact value of:

$$\sin\left(\frac{2\pi}{3}\right)$$

The sine of θ is equal to the y-coordinate of the point of intersection of the terminal side of the angle and the unit circle. Therefore, we need to find the point where the terminal side of this angle here, $\frac{2\pi}{3}$ intersects the unit circle.



That is, we need to find this point here.

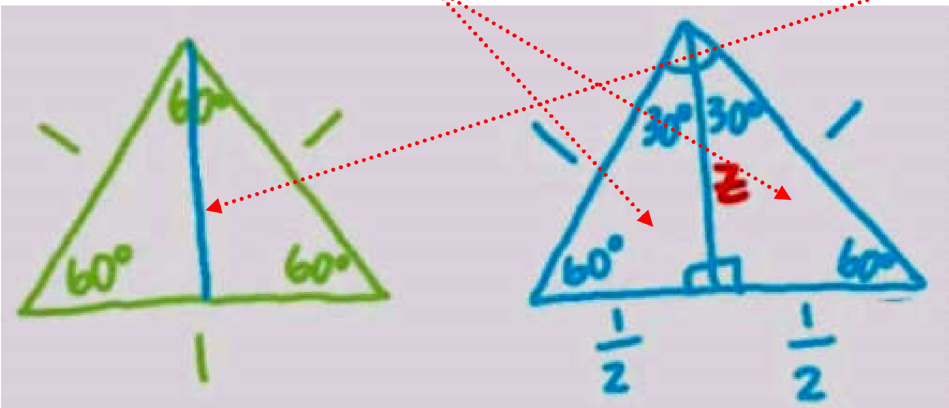
Now, since $\frac{2\pi}{3}$ is in quadrant 2, its reference angle here is:

$$\pi - \frac{2\pi}{3} = \frac{\pi}{3} = 60°$$

Therefore, we can form this triangle as shown. It is a 30°, 60°, 90° triangle with hypotenuse = 1.

Remember, the radius of the unit circle is 1. Therefore, the side adjacent to the 60° angle has length ½, and the side opposite the 60° angle will have the length of $\sqrt{\frac{3}{2}}$.

Let us recall why this is true. Let us draw an equilateral triangle with side lengths = 1. All the angles here are 60°, and then we can split this equilateral triangle into 2 congruent 30°, 60°, 90° triangles by dropping this perpendicular bisector here, that is, we can form these two triangles here.

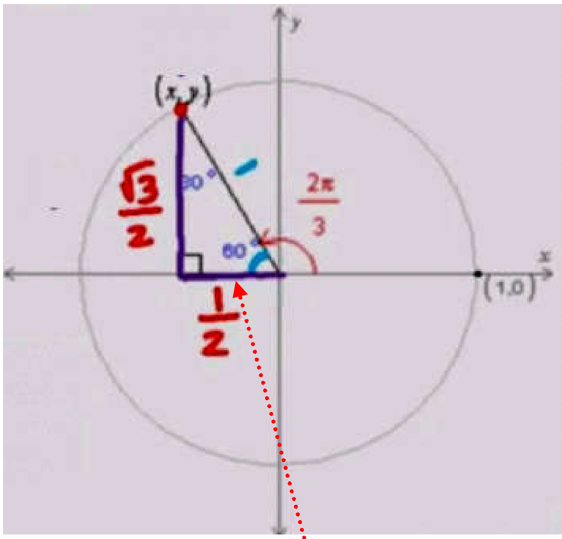


If the total measure of this top angle was 60°, then each of these are 30°, this is still 60°, this still has the length 1, this still has the length 1, but then this bottom side has been split into two pieces of equal length, therefore, both has to be ½.

The question is, what is the length of *z*, of that side of each of those triangles? Well, let us look at this right triangle. By the Pythagorean Theorem, we have that:

$$z^2 + \left(\frac{1}{2}\right)^2 = 1^2$$
$$z^2 = 1 - \frac{1}{4}$$
$$z^2 = \frac{3}{4}$$
$$z = \pm\sqrt{\frac{3}{4}}$$
$$\underline{z = \frac{\sqrt{3}}{2}}$$

But *z* is the length of a side of a triangle, and so we are going to choose the +value.



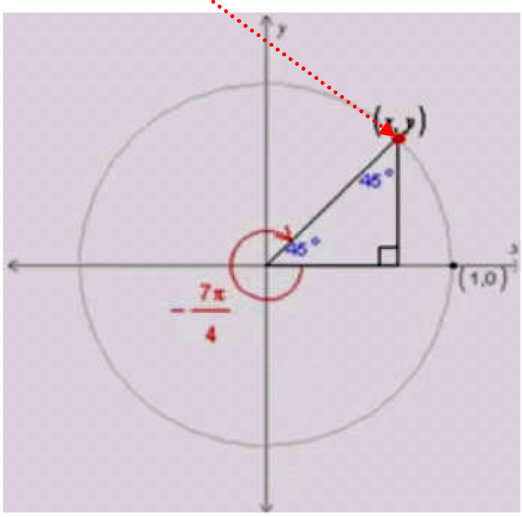
That is, this right triangle here is the triangle we see here.

Now we have the distances $\frac{\sqrt{3}}{2}$ and ½. Remember that (x, y) is in quadrant 2, which means the x-coordinate has to be negative. Therefore, the point then is (-½), the y-coordinate is positive, the y-coordinate is $\frac{\sqrt{3}}{2}$.And therefore, as we said earlier, the sine is the y-coordinate of the point of intersection of the terminal side of the angle and the unit circle.

Alright. Let us look at another example. Let us find the value of:

$$\cos\left(-\frac{7\pi}{4}\right)$$

Now, the cosine of θ is the x-coordinate of the point of intersection of the terminal side of the angle and the unit circle, which means we need to find the point where the terminal side of this angle here, $-\frac{7\pi}{4}$, intersects the unit circle. That is, we need to find this point here.

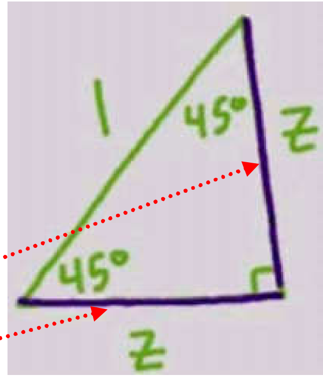


Since $-\frac{7\pi}{4}$ is here in quadrant 1, its reference angle here is:

$$2\pi - \frac{7\pi}{4} = \frac{\pi}{4} = 45^\circ$$

We can draw this triangle shown. It is a 45°, 45°, 90° triangle with hypotenuse = 1, and therefore, the lengths of the other two sides are both $\frac{\sqrt{2}}{2}$.

Again, let us think about why this is true.



You recall from geometry when you have a 45°, 45°, 90° triangle, it forms an isosceles triangle. Let us call this z and this z. Both of these lengths are equal, and by the Pythagorean Theorem we get:

$$\begin{aligned} Z^2 + Z^2 &= 1 \\ 2Z^2 &= 1 \\ Z^2 &= \frac{1}{2} \\ Z &= \pm\sqrt{\frac{1}{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

However, since z represents these two lengths above, then we are going to choose the positive value. Opposite and adjacent (which are isosceles) of the shown angle have the same lengths.

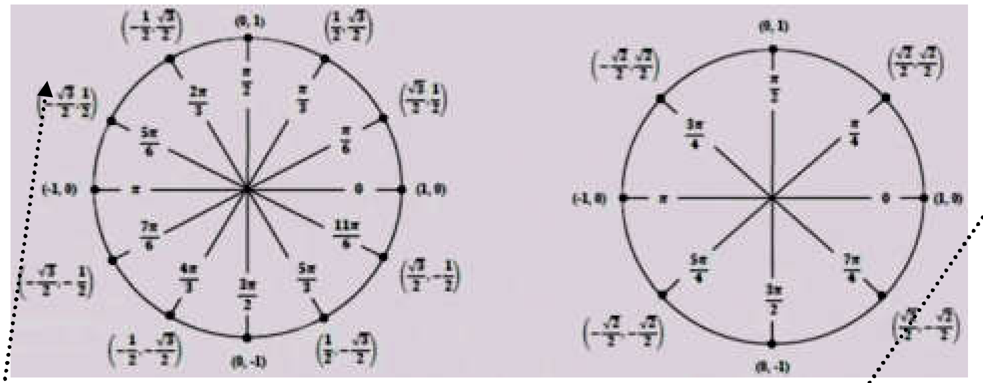
All right. Therefore, the point (x, y) above is:

$$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

Like we said earlier, remember that the cosine of θ is the x-coordinate, the point of intersection of the terminal side of the angle in the unit circle.

7.1 Common Trigonometric Angles

Now, angles whose radian measures are multiples of either $\theta/_6$ $\theta/_4$, like these 2 examples here, are called common trigonometric angles.



This first circle here shows the multiples of $\theta/_6$. And then look, $^4\theta/_6$ or $^2\theta/_3$ was our first example, and remember, we found this point, and then this second circle here shows us the multiples of $\theta/_4$. And remember, the angle we saw in our last example $^{-7}\theta/_4$, which is co-terminal with $\theta/_4$, and we found this point here.

Now, rather than having to derive the x and y-coordinates of these points each time, like we did in our examples, you are going to want to memorize these circles. And really, if we just memorized these points, then, because of the symmetry of the unit circle, it is easy to find the other ones by affixing the appropriate sine in front of the numbers.

For example:

We know that the point that corresponds to $\theta/_6$ is $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. By the symmetry of the unit circle in this angle $^5\theta/_6$ also has the same values $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, but we put a - on the x coordinate, because x is - in quadrant 2.

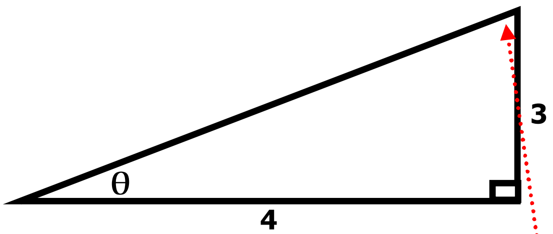
So it will be to your benefit to memorize these circles.

8 Right Triangle Trigonometry

For example, let us find:

$$\sin\theta, \cos\theta, \tan\theta$$

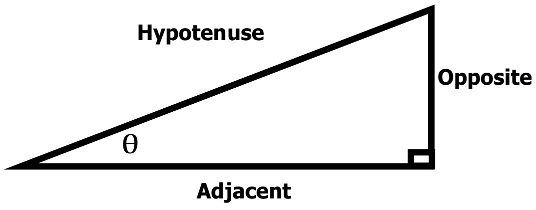
Where θ is the angle shown in this triangle here.



Now, whenever we have a right triangle, and one of the other angles is known, then that determines the other angle, because all the angles have to add up to 180°, which means this other angle here would have to be 90° - θ.

Therefore, the shape of the triangle is completely determined, except for similarity, by the angles. Therefore, once the angles are known, the ratios of the sides are determined, regardless of the overall size of the triangle. In addition, we have names for these ratios.

However, before stating what these ratios are, let us label or give names to these sides of this triangle here.



The side that is across from θ, or opposite θ, we will label as opposite. The side across from the right angle is called the hypotenuse, and the third side next to θ, or adjacent to θ, we call it the adjacent side.

Now we are ready to name these ratios. We have that:

$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$

There is a mnemonic that is commonly used to help us remember these ratios, and the mnemonic is as follows:

SOH CAH TOA

The S here stands for sine, the O stands for opposite and the H stands for hypotenuse. Therefore, we have S, O and H. That is the sin(θ) is = $^{\text{opposite}}/_{\text{hypotenuse}}$. The C here in CAH stands for cosine, the A stands for adjacent and the H stands for hypotenuse; cos(θ) is = $^{\text{adjacent}}/_{\text{hypotenuse}}$. With the TOA, the T stands for tangent, the O stands for opposite and the A stands for adjacent; we have tan(θ) is = $^{\text{opposite}}/_{\text{adjacent}}$.

With this background information, we can solve our problem. We can now find sin(θ), cos(θ), and tan(θ). Looking at our triangle we have θ, O = 3 and A = 4. However, what is the hypotenuse, H?

Well, the hypotenuse we can find by using the Pythagorean Theorem, namely:

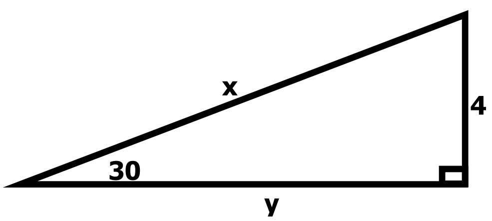
$$3^2 + 4^2 = H^2$$
$$9 + 16 = H^2$$
$$H^2 = 25$$
$$\underline{H = 5}$$

Therefore, coming back over to these ratios, the sin(θ) is = $^{\text{opposite}}/_{\text{hypotenuse}}$, the cos($\theta$) is = $^{\text{adjacent}}/_{\text{hypotenuse}}$, and the tan($\theta$) is = $^{\text{opposite}}/_{\text{adjacent}}$. So, these would be our three answers here.

$$\sin\theta = \frac{3}{5}$$
$$\cos\theta = \frac{4}{5}$$
$$\tan\theta = \frac{3}{4}$$

By the way, the word trigonometry comes from the Greek words meaning triangle and measure, and you should see now why.

All right, let us look at another example. Let us find *x* and *y* in this figure here.



Let us start by labeling our given angle, θ = 30°. The 4 is the length of the side opposite θ; let us label it O. *y* down there is the length of the side that is adjacent to θ; we will label it A. *x* up there is the hypotenuse; we will label it H.

By what we just learned:

$$\sin 30^\circ = \frac{O}{H} = \frac{4}{x}$$
$$\cos 30^\circ = \frac{A}{H} = \frac{y}{x}$$
$$\tan 30^\circ = \frac{O}{A} = \frac{4}{y}$$

Now, we can use this first equation to solve for *x*.

$$\sin 30^\circ = \frac{1}{2}$$
$$\frac{4}{x} = \frac{1}{2}$$
$$\underline{x = 8}$$

Now, to solve for *y* we can use either of these last 2 equations. That is, we could use this middle equation.

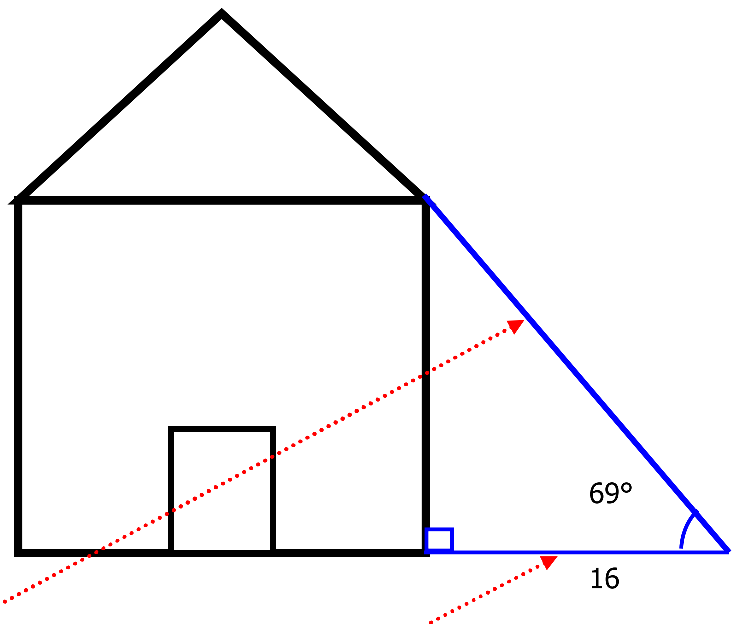
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$\frac{\sqrt{3}}{2} = \frac{y}{8}$$
$$\underline{y = 4\sqrt{3}}$$

Now, this computation though required that we knew what this value of *x* was. If we use this third equation though, we did not need to know what *x* was.

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$
$$\frac{1}{\sqrt{3}} = \frac{4}{y}$$
$$\underline{y = 4\sqrt{3}}$$

9 Right Triangle Word Problems

For example, a ladder leans against the side of a house. If the angle of elevation of the ladder is 69°, when the bottom of the ladder is 16 feet from the side of the house let us find the length of the ladder.



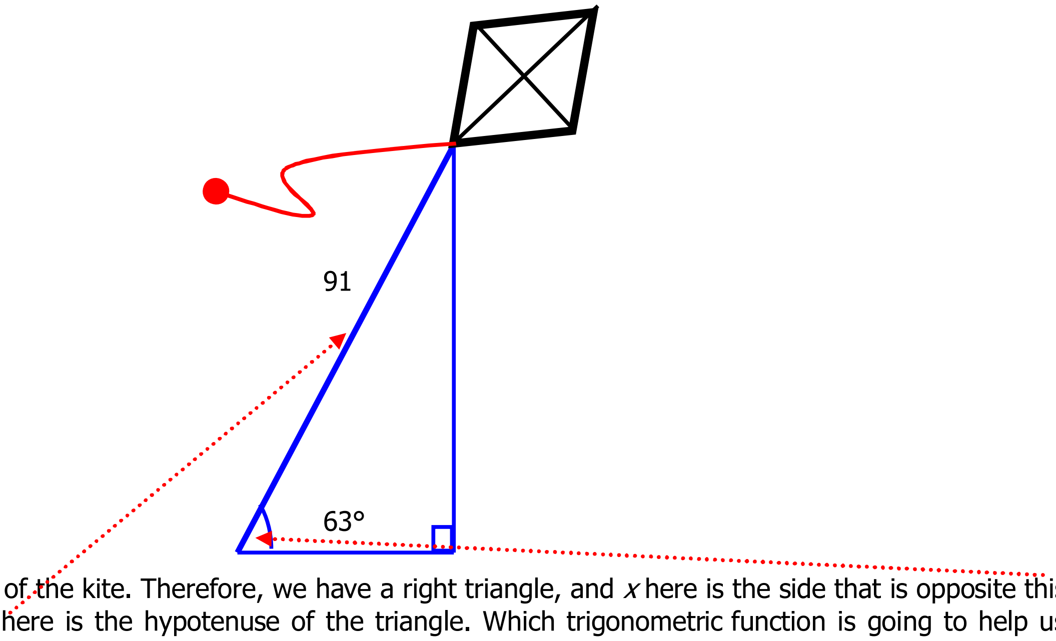
Let us say x = that length. Now, how we can solve for *x*?

Well, we have a right triangle and 16 here, this is the side adjacent to the given triangle, and *x* is the hypotenuse. Therefore, the cosine function can help us. Therefore, for our triangle here, that means that:

$$\begin{aligned}\cos\theta &= \frac{adjacent}{hypotenuse}\\ \cos 69^\circ &= \frac{16}{x}\\ x &= \frac{16}{\cos 69^\circ}\\ &\approx \underline{44.7 feet}\end{aligned}$$

Let us look at another example.

A kite flying in the air has a 91 ft line attached to it, and the line is pulled taut. The angle of elevation of the kite is 63°. Find the height of the kite.



Let x = the height of the kite. Therefore, we have a right triangle, and x here is the side that is opposite this angle of 63° and 91 here is the hypotenuse of the triangle. Which trigonometric function is going to help us here? Would that not be sine?

Because sin(θ), remember, is = $^{opposite}/_{hypotenuse}$. Therefore, in our situation we have:

$$\begin{aligned}\sin\theta &= \frac{opposite}{hypotenuse}\\ \sin 63^\circ &= \frac{x}{91}\\ x &= 91 * \sin 63\\ &\approx \underline{81.1 feet}\end{aligned}$$

These are a few examples of right triangle word problems.

Properties of Trigonometric Functions, Inverse Trigonometric Functions

1 Trigonometric Values given a Point on the Unit Circle

For example, let us suppose that θ is an angle in standard position, whose terminal side intersects the unit circle at:

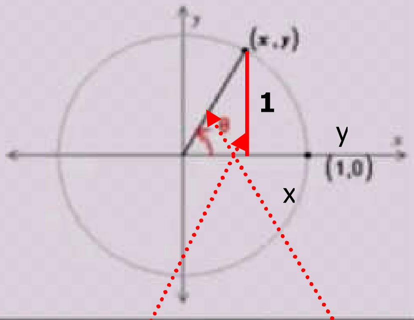
$$\left(-\frac{21}{29}, -\frac{20}{29}\right)$$

Let us find the exact values of $\sin(\theta)$, $\tan(\theta)$ and csc[3](θ). We define the trigonometric functions in terms of coordinates of points on the unit circle as follows. Suppose that θ is an angle in standard position, whose terminal side intersects the unit circle at (x, y), then the six trigonometric functions are defined as follows.

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y},\ y \neq 0$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$



Now, these definitions are consistent with the right triangle trigonometric ratios that you might be used to. To see this, suppose that θ is acute like the angle shown here.

Let us drop a perpendicular from the point (x, y) to the x-axis. Now, this length then is y, and this is x, and remember, the radius of this circle is 1, therefore, the length is 1. Using our trigonometric ratios, we have:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{1} = y$$

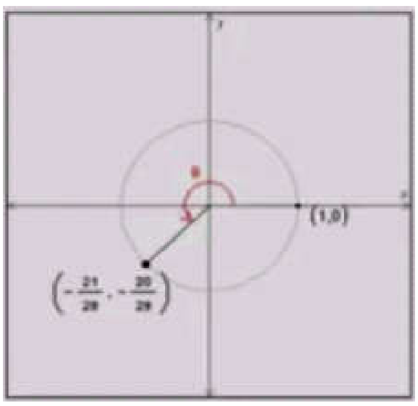
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{1} = x$$

Okay. So let us apply this to our situation here. We are given that:

$$x = \frac{-21}{29}$$

$$y = \frac{-20}{29}$$

as shown in this figure.



[3] cosecant

Let us use this definition to find sin(θ), tan(θ) and csc(θ). Namely:

$$\sin\theta = y$$
$$= \frac{-20}{29}$$

$$\tan\theta = \frac{y}{x}$$
$$= \frac{\frac{-20}{29}}{\frac{-21}{29}}$$
$$= \frac{20}{21}$$

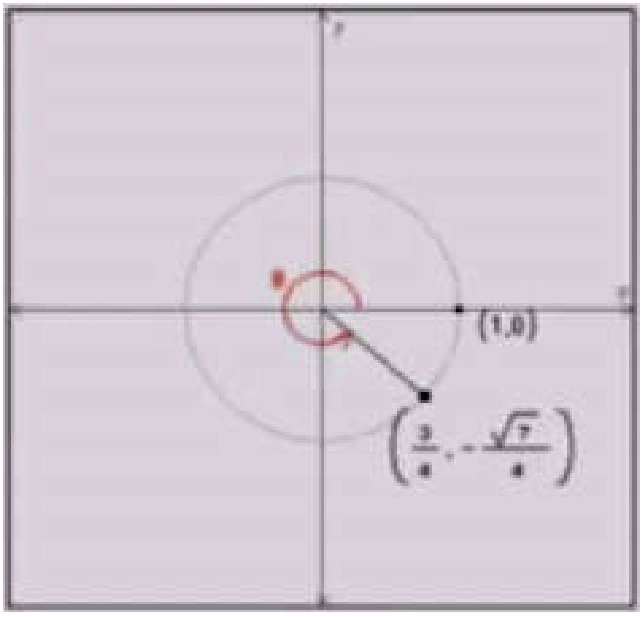
$$\csc\theta = \frac{1}{y}$$
$$= \frac{1}{\frac{-20}{29}}$$
$$= \frac{-29}{20}$$

Therefore, these are the three values that we were looking for sin(θ), tan(θ) and csc(θ).

Alright. Let us look at another example. Suppose that θ is an angle in standard position whose terminal side intersects the unit circle at:

$$\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$$

Let us find the exact values of cos(θ), cot(θ) and sec(θ). Again, we are going to use the following definition, but now we are going to find cos(θ), cot(θ) and sec(θ). So we are given that the terminal side of θ intersects the unit circle as shown in the figure here.



The coordinates at this point are:

$$\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$$

Therefore:

$$\begin{aligned}\cos\theta &= x\\ &= \frac{3}{4}\end{aligned}$$

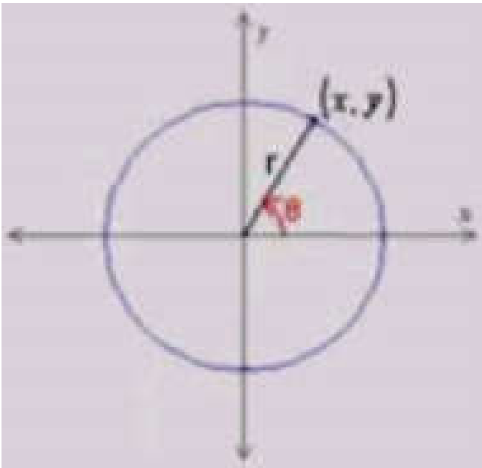
$$\begin{aligned}\cot\theta &= \frac{x}{y}\\ &= \frac{\frac{3}{4}}{\frac{-\sqrt{7}}{4}}\\ &= -\frac{3}{\sqrt{7}} * \frac{\sqrt{7}}{\sqrt{7}}\\ &= -\frac{3\sqrt{7}}{7}\end{aligned}$$

$$\begin{aligned}\sec\theta &= \frac{1}{x}\\ &= \frac{1}{\frac{3}{4}}\\ &= \frac{4}{3}\end{aligned}$$

Therefore, these are the three values we are looking for.

2 Trigonometric Values given Information about the Angle

For example, let (12, -5) be a point on the terminal side of θ Find the exact values of cos(θ), cosec(θ), and tan(θ).



Now, the first thing to notice is that this point does not lie on the unit circle, because in order for a point to lie on the unit circle:

$$\begin{aligned}x2 + y2 &= 1\\ (12)^2 + (-5)^2 &= 144 + 25\\ &= 169 \neq 1\end{aligned}$$

Therefore, we cannot use the unit circled definition of the trigonometric functions, but there is an equivalent one.

That is, if (x, y) is the point of intersection of the *terminal side* of the angle and a circle with radius *r*, then the *trigonometric functions* are given as follows:

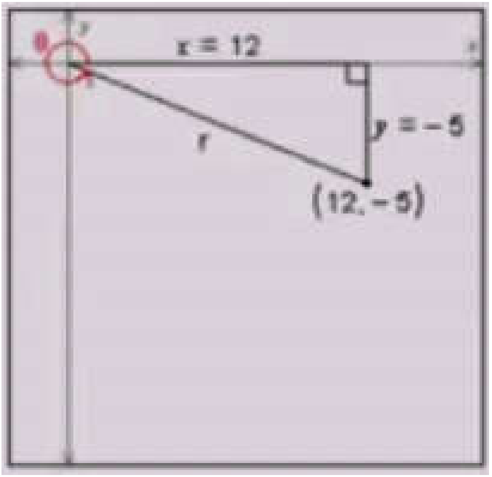
$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, x \neq 0$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$

Therefore, in our case, we have (x ,y) = (12, -5).



We will find *r* by using:

$$\begin{aligned}(x, y) &= (12, -5)\\ r &= \sqrt{x^2 + y^2}\\ &= \sqrt{(12)^2 + (-5)^2}\\ &= \sqrt{144 + 25}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

Now that we know *r*, we can find the 3 values here by using the equivalent equation from above.

$$\cos\theta = \frac{x}{r} = \frac{12}{13}$$

$$\csc\theta = \frac{r}{y} = \frac{-13}{5}$$

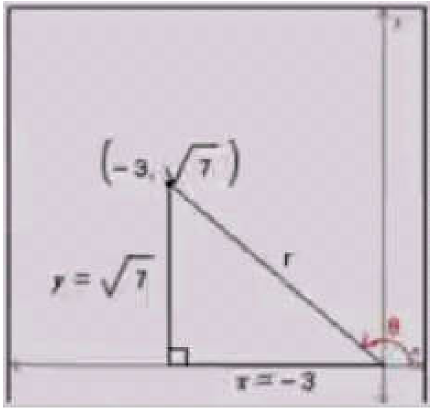
$$\tan\theta = \frac{y}{x} = \frac{-5}{12}$$

Let us look at another example. Let $\left(-3, \sqrt{7}\right)$ be a point on the terminal side of θ. Find the exact values of sin(θ), sec(θ) and cot(θ). Again, our given point here does not lie on the unit circle, because:

$$\begin{aligned}(-3)^2 + \left(\sqrt{7}\right)^2 &= 9 + 7\\ &= 16 \neq 1\end{aligned}$$

Therefore, we cannot use the unit circle definitions of the trigonometric functions. We are going to use the definitions of trigonometric functions instead.

We have that $(x, y) = \left(-3, \sqrt{7}\right)$ as shown in this figure.



Let us find *r* by using above mentioned equation again:

$$
\begin{aligned}
r &= \sqrt{(-3)^2 + \left(\sqrt{7}\right)^2} \\
&= \sqrt{9 + 7} \\
&= \sqrt{16} \\
&\underline{= 4}
\end{aligned}
$$

Notice; in both of these examples we are basically doing the same work 2*, because looking up here, did not we already know that $r^2 = 16$? So basically you would just have to take the square root of that number to get your *r* each time.

Alright, now we are ready to find the sin(θ), sec(θ) and cot(θ) by the above mentioned formulas.

$$\sin\theta = \frac{y}{r} = \underline{\frac{\sqrt{7}}{4}}$$

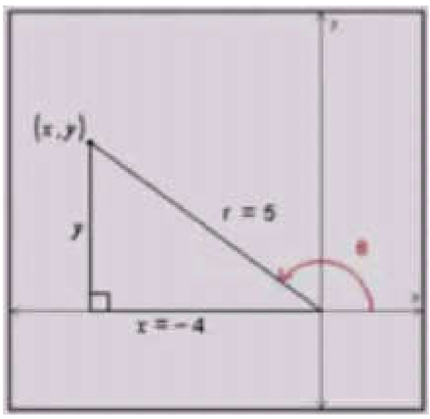
$$\sec\theta = \frac{r}{x} = \underline{\frac{-4}{3}}$$

$$
\begin{aligned}
\cot\theta &= \frac{x}{y} = \frac{-3}{\sqrt{7}} \\
&= \frac{-3}{\sqrt{7}} * \frac{\sqrt{7}}{\sqrt{7}} = \underline{\frac{-3\sqrt{7}}{7}}
\end{aligned}
$$

Let us find trigonometric values given information about the angle.

For example, let θ be an angle such that $\cos\theta = -\frac{4}{5}$ and $\sin\theta > 0$. Find the exact values of the other 5 trigonometric functions.

Now, $\cos\theta$ is negative. Then *q* is either in quadrant 2 or quadrant 3, because cosine values are - in those two quadrants. However, we are also told that $\sin\theta > 0$, and sine values are not positive in quadrant 3, which means we can conclude that θ is in quadrant 2, and we can draw the following reference triangle.



Now we will need to find y before we start computing the asked values.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-4)^2 + y^2 &= (5)^2 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \\ \underline{y} &\underline{= \pm 3} \end{aligned}$$

As we did not have negative lengths in a triangle, we choose $y = 3$. Now remember our trigonometric function definitions!

$$\cos\theta = \frac{x}{r} = \underline{\frac{-4}{5}}$$

$$\sin\theta = \frac{y}{r} = \underline{\frac{3}{5}}$$

$$\tan\theta = \frac{y}{x} = \underline{\frac{-3}{4}}$$

$$\csc\theta = \frac{r}{y} = \underline{\frac{5}{3}}$$

$$\sec\theta = \frac{r}{x} = \underline{\frac{-5}{4}}$$

$$\cot\theta = \frac{x}{y} = \underline{\frac{-4}{3}}$$

3 Inverse Trigonometric Values

For example, let us find the exact value of:

$$\cos^{-1}\left(\frac{1}{2}\right)$$

Now, be very careful here with this notation. It does not mean:

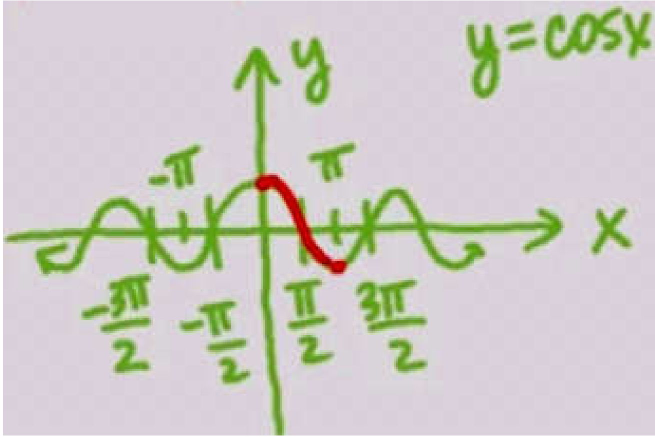
$$\cos^{-1}(x) \neq \frac{1}{\cos x}$$

This is just the notation for the inverse cos function. If we meant this, we would have write $[\cos x]^{-1}$. By the way, there is another notation for the inverse cos function. That avoids this ambiguity.

$$\cos^{-1} x = \arccos x$$

However, this notation is more common.

What does this mean, inverse cos function? If we think of the graph of cos, therefore, cos is not a 1 : 1-function, is it?



This is the graph of y = cos(x). Remember that in order for a function to have an inverse, it must be 1 : 1. In order to discuss this function here, this inverse cos function what we do is we restrict the domain of cos.

$$Dom = [0, \pi]$$

If we restrict the domain as shown above, then cos will be 1 : 1 on this interval. Therefore, we restrict the domain of cos to this given interval zero. Remember that the domain of a function is the range of the inverse. Therefore, we have the following definitions.

Function	Meaning
Inverse Sine Function	$y = \sin^{-1} x$ means that $\sin y = x$ and y is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Inverse Cosine Function	$y = \cos^{-1} x$ means that $\cos y = x$ and y is in $[0, \pi]$
Inverse Tangent Function	$y = \tan^{-1} x$ means that $\tan y = x$ and y is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

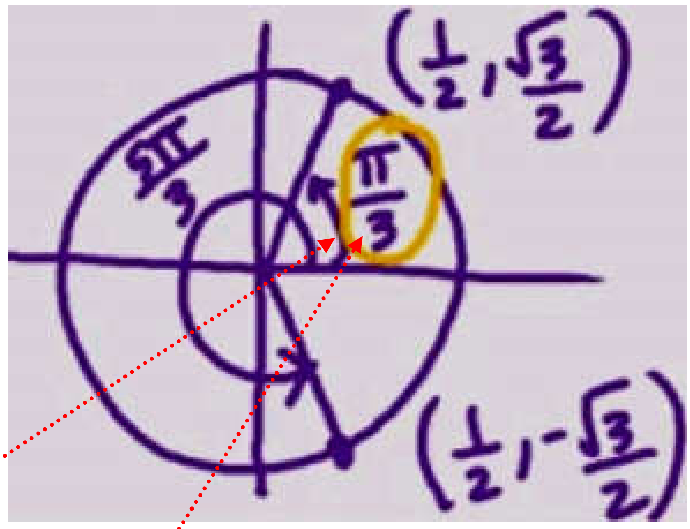
We are working with this middle one, the inverse cos function. This is a restricted domain of our cos function, which is the range of our inverse function. So we are thinking backwards now.

Our inverse cos values will be angles. We are used to plugging in angles into the cos function; now they are the output.

$$\cos^{-1}\left(\frac{1}{2}\right) \Leftrightarrow \cos y = \frac{1}{2}$$

Where is y in our interval?

Thinking of our unit circle,

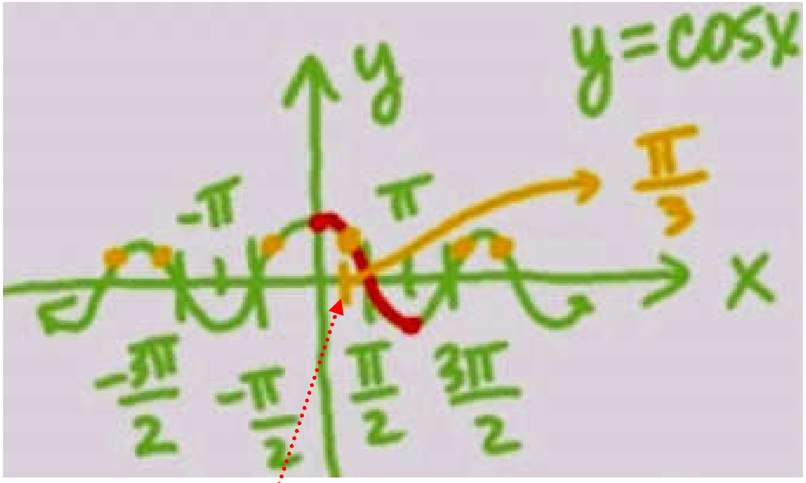


the cos of this angle here is shown here.

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$y = \cos^{-1}\left(\frac{1}{2}\right) \Leftrightarrow \cos y = \frac{1}{2}$$

However, the only angle in this interval is $\frac{\pi}{3}$. Which would be our answer?



As we look above at the green graph, this angle here is $\frac{\pi}{3}$, and the cos of it will be = ½. Also, all the y-coordinates of these points is also = ½. This means that the cos of all of those angles, which corresponds to these points, will be ½ well.

Alright, let us look at another example.

Let us find the exact value of:

$$\sin^{-1}\left(-\frac{1}{2}\right)$$

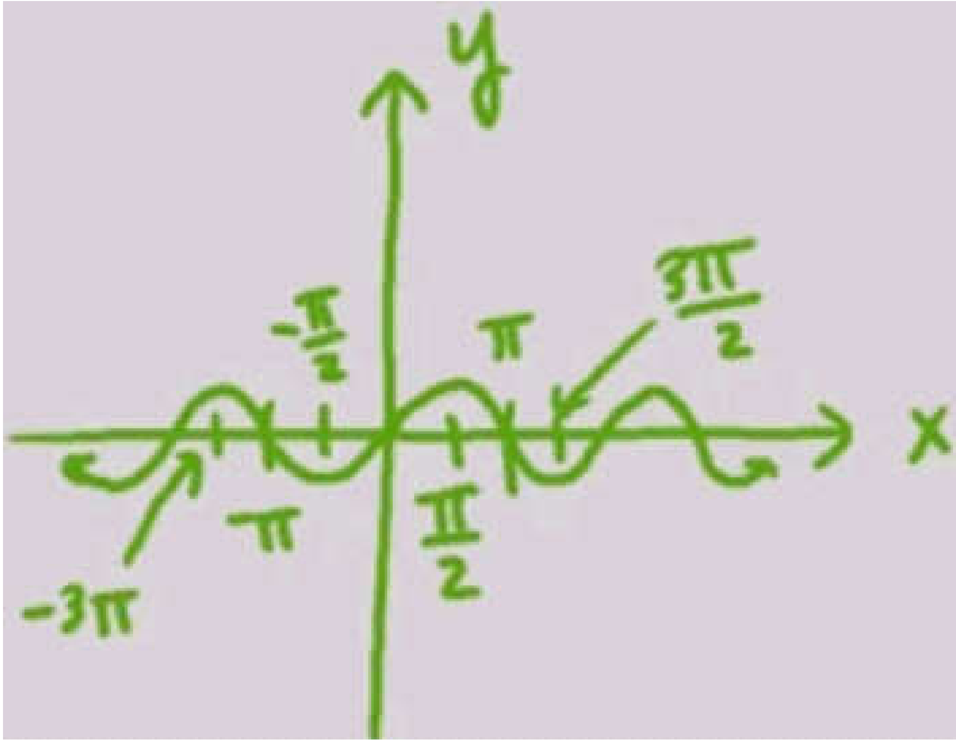
Again, we have the following definitions.

Function	Meaning
Inverse Sine Function	$y = \sin^{-1} x$ means that $\sin y = x$ and y is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Inverse Cosine Function	$y = \cos^{-1} x$ means that $\cos y = x$ and y is in $[0, \pi]$
Inverse Tangent Function	$y = \tan^{-1} x$ means that $\tan y = x$ and y is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y = \sin^{-1}\left(-\frac{1}{2}\right) \Leftrightarrow \sin y$$

$$Dom = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Let us graph this sin function to see why this interval works.



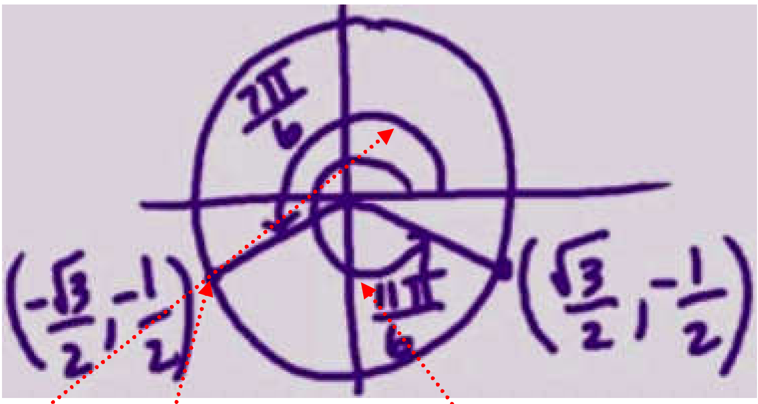
Moreover, again, y = sin(x) is not a 1 : 1-function. However, on the interval from $^{-}\pi/_{2}$ to $-\pi/_{2}$, for example, it is 1 : 1, is it not?

On that restricted domain, sin will be 1 : 1, and therefore, it will have an inverse. Remember the restricted domain is now the range of our inverse, and notice that is a different interval than what we just saw with the cos function.

Alright, so let us compute our value.

$$y = \sin^{-1}\left(-\frac{1}{2}\right) \Leftrightarrow \sin y = -\frac{1}{2}$$

Thinking of our unit circle we find:



The sin of this angle here is

$$\frac{7\pi}{6} = -\frac{1}{2},$$

because the y-coordinate of this point is -½, but also this angle here, which corresponds to $^{11}\pi/_{6}$ also has sin = -½, because the y-coordinate of this point is also -½. The sin of any of their co-terminal angles will also be -½. However, looking in the table, what we need to do is find the angle in this interval that has sin = -½.

Now, a very common mistake that students will make is, they will think that the interval, which is our domain, corresponds to any angle in quadrants 4 and 1, and therefore, they will answer this $^{11}\pi/_{6}$.

However, $^{11}\pi/_{6}$ does not lie in our domain. What we need to do is find the angle that is co-terminal with $^{11}\pi/_{6}$ whose sign is = --½. Moreover, that angle would be $^{-}\pi/_{6}$.

Let us look at that on our graph.

This angle here is $^{-\pi}/_6$. The sin of it will be -½, but also the y-coordinates of all of these points here are -½, which means that the sin of all of those angles corresponding to those points will also be = -½, but the only one in the restriction is $-^{\pi}{}_6$. Therefore, we have to be really careful when computing inverse sin-values of a negative input, and this is the same thing with computing inverse tangent of -inputs.

So let us see an example of that. Let us find the exact value of:

$$y = \tan^{-1}(-1)$$

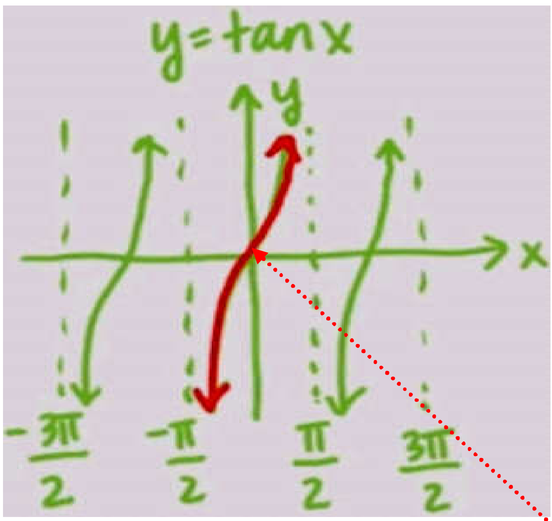
Again we have the following definitions and we will be working with the last 1.

Function	Meaning
Inverse Sine Function	$y = \sin^{-1} x$ means that $\sin y = x$ and y is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Inverse Cosine Function	$y = \cos^{-1} x$ means that $\cos y = x$ and y is in $[0, \pi]$
Inverse Tangent Function	$y = \tan^{-1} x$ means that $\tan y = x$ and y is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Therefore:

$$y = \tan^{-1}(-1) = \tan y = -1$$
$$Dom = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

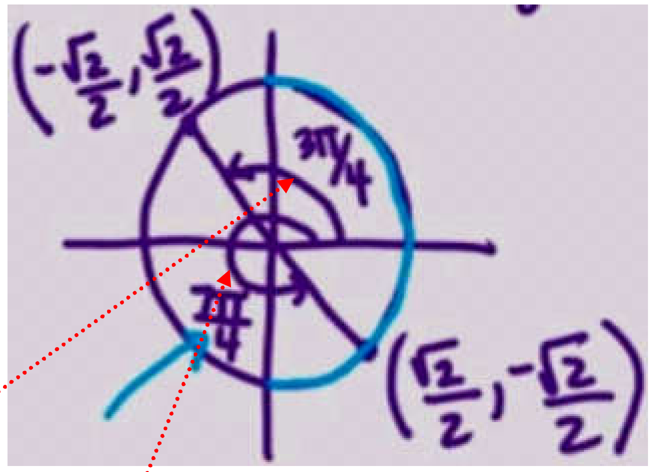
Let us take a look at the graph of tangent to see why this interval works. Now $y = \text{tang}(x)$ looks like this.



Within each of these consecutive vertical asymptotes, tangent looks like this. Notice that tangent is not a 1 : 1-function, and therefore, it will not have an inverse on its entire domain. However, it will have an inverse if we restrict its domain.

On this restricted domain the tangent function will be 1 : 1, and the domain of the tangent function will then be the range of the inverse. Therefore, our answers to inverse tangent values have to lie in that interval (see above).

Again, let us think of our unit circle.



This angle here, $^{3\pi}/_{4}$, will have tan = -1, because the ratio of the y-coordinate to the x-coordinate will be -1, but also the tangent of this angle here, $^{7\pi}/_{4}$ be -1. Because the ratio of the y-coordinate to the x-coordinate here is -1, and the tangent of any of their co-terminal angles is also = -1, But remember, looking at our domain we have to have our answer in this interval, and, again, be very careful here. Do not just look in quadrant 4 and 1, and be tempted to answer $^{7\pi}/_{4}$. We have to find the angle that is co-terminal with $^{7\pi}/_{4}$, but that lies in this interval down there, and that angle would be $^{-\pi}/_{4}$, would not it?

Our answer then is:

$$y = \tan^{-1}(-1) = \frac{-\pi}{4}$$

4 Compositions Involving Inverse Trigonometric Functions

For example, let us find the exact value of:

$$\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$$

$$y = \cos^{-1}\frac{\sqrt{3}}{2}$$

$$\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \tan y$$

So we are looking for tan(y):

$$y = \cos^{-1}\frac{\sqrt{3}}{2}$$

$$\cos y = \frac{\sqrt{3}}{2} \quad \textit{where } y \textit{ is in } [0,\pi]$$

Remember, that when computing inverse cosine-values, our answers must lie on that interval. Thinking of our unit circle, and remembering that the cosine is the x-coordinate of the point of intersection of the terminal side of the angle and the unit circle, and we know that the terminal side of the angle $^{\pi}/_{6}$, intersects the unit circle at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore, $\cos y = \frac{\sqrt{3}}{2}$, and, moreover, it is the only angle in this interval, whose cosine is $= \frac{\sqrt{3}}{2}$. That is y is = $^{\pi}/_{6}$.

Therefore:

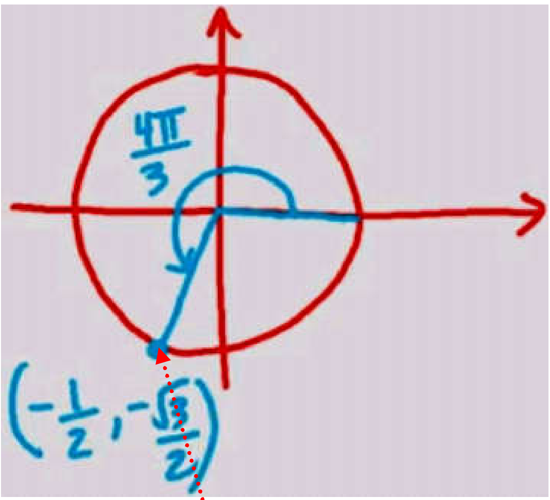
$$\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \tan y$$
$$= \tan\left(\frac{\pi}{6}\right)$$
$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$
$$= \frac{1}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

Alright, let us look at another example. Let us find the exact value of:

$$\arcsin\left(\cos\frac{4\pi}{3}\right)$$

Moreover, what is cosine of $^{4\pi}/_{3}$?

Well, thinking of our unit circle, the terminal side of the angle $^{4\pi}/_{3}$ intersects a unit circle at:



The cosine is the x-coordinate of this point here is -½. So therefore this is:

$$\arcsin\left(\cos\frac{4\pi}{3}\right) = \arcsin\left(-\frac{1}{2}\right)$$
$$= -\frac{\pi}{6}$$
$$y = \arcsin\left(-\frac{1}{2}\right) \Leftrightarrow \sin y = -\frac{1}{2}$$
$$y \textit{ is in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Thinking again of the unit circle, we know that the terminal side of this angle, which is $^{7\pi}/_{6}$, intersects the unit circle at:

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

In addition, the terminal side of the angle $^{11\pi}/_6$ intersects the unit circle at:

$$\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

which means, that the sine of both of these angles is = -½, because remember that the sine is the y-coordinate of these points. However, neither of these angles lies within the restriction of arcsin. So what we need to do is determine the angle that is co-terminal with $^{11\pi}/_6$, lying in that interval, which is $^{-\pi}/_6$.

So this is our answer.

$$\arcsin\left(\cos\frac{4\pi}{3}\right)=\underline{-\frac{\pi}{6}}$$

Alright, let us look at another example. For example, let us find the exact value of:

$$\tan\left(\arccos\left(-\frac{5}{6}\right)\right)$$

Now, this is an angle, so let us call it:

$$\tan\left(\arccos\left(-\frac{5}{6}\right)\right)\qquad \textit{with}\qquad \theta=\arccos\left(-\frac{5}{6}\right)$$

$$\tan\theta$$

If

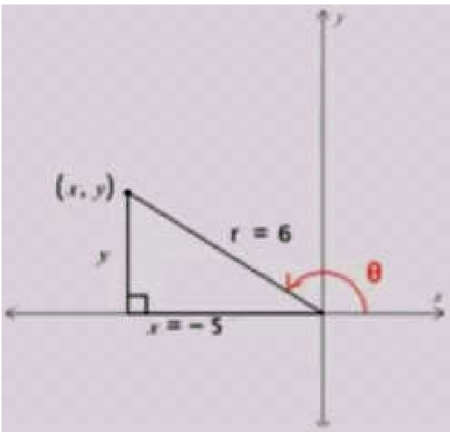
$$\theta=\arccos\left(-\frac{5}{6}\right)$$

$$\cos\theta=-\frac{5}{6}\qquad \&\qquad \theta \textit{ is in } Q2$$

Now, why is θ in quadrant 2? Well, looking here, we see that:

$$\cos\theta<0\qquad \textit{either }\ \theta \textit{ is in } Q2 \textit{ or } \cancel{Q3}$$

However, remember looking back up here, this means that θ also has to be in the range of an arccos-function, which, remember, the range of arccos is the interval $[0, \pi]$. Therefore, θ cannot be in quadrant 3, which means, θ is in quadrant 2, and, therefore, we can draw the following triangle.



Because remember, that:

$$\cos\theta=\frac{x}{r}$$
$$=\frac{-5}{6}$$

In addition, notice over here, we are putting the –x, because x is negative in quadrant 2, and *r* is always positive. Therefore, by the Pythagorean Theorem, we can find *y*.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-5)^2 + y2 &= (6)^2 \\ 25 + y^2 &= 36 \\ y^2 &= 11 \\ \underline{y} &\underline{= \pm\sqrt{11}} \end{aligned}$$

However, which one do we choose, the + or the -? Remember that *y* is in quadrant 2 here, and in quadrant 2, y > 0. Therefore, we are going to choose the +value.

Now remember, looking back up, we are looking for tangent of θ, which is equal to the y-coordinate divided by the x-coordinate, which is:

$$\tan\left(\arccos\left(-\frac{5}{6}\right)\right)$$

$$\textit{with} \quad \theta = \arccos\left(-\frac{5}{6}\right)$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{\sqrt{11}}{-5} \\ &\underline{= \frac{-\sqrt{11}}{5}} \end{aligned}$$

Basic Trigonometric Identities: Sum, Difference, Co-Functions, Double-Angle and Half-Angle

1 Verifying Trigonometric Identities

For example, let us verify this identity here.

$$\frac{1-\cos^2 y}{(1-\sin y)(1+\sin y)}=\tan^2 y$$

Now while verifying trigonometric identities we can either work with 1 side, until we get it equal to the other side, or work with both sides at the same time, until we get them equal to the other. So in this case, the left-hand side looks a little bit more complicated. So let us start working with that side first.

$$\frac{1-\cos^2 y}{(1-\sin y)(1+\sin y)}=\tan^2 y$$

$$\begin{aligned}\frac{1-\cos^2 y}{(1-\sin y)(1+\sin y)}&=\frac{1-\cos^2 y}{1+\sin y-\sin y-\sin^2 y}\\&=\frac{1-\cos^2 y}{1-\sin^2 y}\end{aligned}$$

Now let us recall the following Pythagorean Identity:

$$\begin{aligned}\sin^2 y&=+\cos^2 y=1\\ \sin^2 y&=1-\cos^2 y\\ \cos^2 y&=1-\sin^2 y\end{aligned}$$

So let us use those facts.

$$\frac{1-\cos^2 y}{(1-\sin y)(1+\sin y)}=\tan^2 y$$

$$\begin{aligned}\frac{1-\cos^2 y}{(1-\sin y)(1+\sin y)}&=\frac{1-\cos^2 y}{1+\sin y-\sin y-\sin^2 y}\\&=\frac{1-\cos^2 y}{1-\sin^2 y}\\&=\frac{\sin^2 y}{\cos^2 y}\\&=\underline{\tan^2 y}\end{aligned}$$

All right, let us look at another example: Let us verify this identity here.

$$\csc x-\sin x=\cot x\cos x$$

Now with this identity let us work with both sides at the same time. Recall that:

$$\csc x=\frac{1}{\sin x}$$

$$\cot x=\frac{\cos x}{\sin x}$$

We will use this on our identity. That is:

$$\csc x - \sin x = \cot x \cos x$$
$$\frac{1}{\sin x} - \sin x = \frac{\cos x}{\sin x} * \cos x$$
$$\frac{1-\sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}$$
$$\underline{\frac{\cos^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}}$$

Therefore, these are equal, and we verified the identity.

2 Sum and Difference Identities

For example, let us find the exact value of

$$\sin\frac{\pi}{12}$$

by using a sum or difference identity.

Well, let us first recall these identities.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Now, since we are looking for a sin value, we will want to use 1 of the first 2 identities here. So what we need to do is find 2 angles, *A* and *B*, which either add or subtract to give us this $\frac{\pi}{12}$. But, moreover, whose sine and cosine values we know.

Now, let us recall some of our common trigonometric angles, $\pi/2$, $\pi/6$, $\pi/4$, $\pi/3$, and so on. Now, we know the sine and cosine of all of these angles. We want to find 2 of them that will add or subtract to give us $\frac{\pi}{12}$. Looking here, these denominators, 4 and 3, multiply together to give us 12, so they seem pretty promising, and, in fact,

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$$

Since we have written our angle $\frac{\pi}{12}$ as a difference, we will be using this difference identity for sine.

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} - \frac{1}{2} * \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \underline{\frac{\sqrt{6}-\sqrt{2}}{4}}\end{aligned}$$

Now, is this the only way we could have written $\frac{\pi}{12}$ as the sum or difference of these common trigonometric angles? Moreover, it is not.

It is not the only break up. In fact, let us compute this:

$$\begin{aligned}\frac{\pi}{4}-\frac{\pi}{6} &= \frac{6\pi-4\pi}{24}\\ &= \frac{2\pi}{24}\\ &= \frac{\pi}{12}\end{aligned}$$

That is, $\frac{\pi}{12}$ can also be written as this difference. This means we could have also computed $\sin\frac{\pi}{12}$ as follows.

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}*\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}*\frac{1}{2}\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Therefore, in some cases, there is more than one way to apply these sum and difference identities.

All rright, let us look at another example: Let us find the exact value of this expression here by using a sum or difference identity.

$$\cos 16^\circ \cos 14^\circ - \sin 16^\circ \sin 14^\circ$$

To compute this value here without using one of these identities would be very challenging, because 16° is not one of our common trigonometric angles, neither is 14°. However, the sum and difference identities will be very helpful here. Let us recall these identities.

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ \sin(A-B) &= \sin A\cos B-\cos A\sin B\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B\\ \tan(A+B) &= \frac{\tan A+\tan B}{1-\tan A\tan B}\\ \tan(A-B) &= \frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}$$

Matching our expression to the right-hand sides of these identities, we see it matches the third one with A = 16° and B = 14°. That is:

$$\begin{aligned}\cos 16^\circ \cos 14^\circ - \sin 16^\circ \sin 14^\circ &= \cos A\cos B-\sin A\sin B\\ &= \cos(A+B)\\ &= \cos(16^\circ+14^\circ)\\ &= \cos(30^\circ)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Using these identities helped us compute this value quite easily.

Let us look at another example. Let us use a sum or difference identity to find the exact value of this expression here.

$$\frac{\tan\frac{\pi}{8}+\tan\frac{5\pi}{8}}{1-\tan\frac{\pi}{8}\tan\frac{5\pi}{8}}$$

Again, $\frac{\pi}{8}$ is not 1 of our common trigonometric values, neither is $\frac{5\pi}{8}$, but the sum and difference identities can help us solve this quite easily. Again, we are going to compare our expression to the right-hand sides, and we see that it matches with the fifth one with A = $\frac{\pi}{8}$, and B = $\frac{5\pi}{8}$.

$$\begin{aligned}\frac{\tan\frac{\pi}{8}+\tan\frac{5\pi}{8}}{1-\tan\frac{\pi}{8}\tan\frac{5\pi}{8}} &= \frac{\tan A+\tan B}{1-\tan A\tan B}\\ &= \tan(A+B)\\ &= \tan\left(\frac{\pi}{8}+\frac{5\pi}{8}\right)\\ &= \tan\left(\frac{6\pi}{8}\right)\\ &= \tan\left(\frac{3\pi}{4}\right)\\ &\underline{=-1}\end{aligned}$$

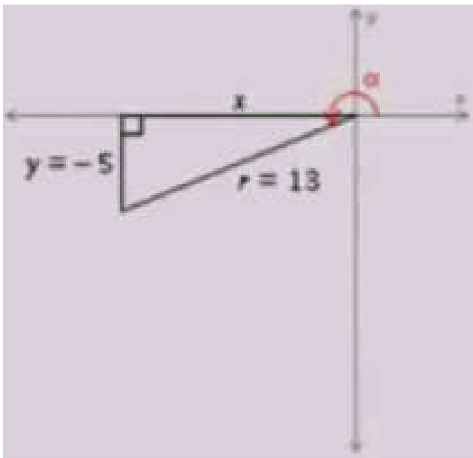
Again, let us look at another example, let us find:

$$\begin{aligned}&\cos(\alpha-\beta)\\ &\text{if } \sin\alpha=-\frac{5}{13},\cos\beta=\frac{12}{13}\\ &\alpha \text{ is in Q3}\\ &\beta \text{ is in Q4}\end{aligned}$$

Let us recall the difference formula for cosine. We have:

$$\cos(\alpha\mp\beta)=\cos\alpha\cos\beta\pm\sin\alpha\sin\beta$$

Remember with the cosine identities, if this is a - then this is a +. Now we are given that $\sin\alpha=-\frac{5}{13}$. Therefore, we know this already, and we are also given $\cos\beta=\frac{12}{13}$, which means we know this as well. So it remains to find cos(β) and sin(α). Now there are different approaches here. We could use identities, for example, but let us use triangles instead. Now we are given, that $\sin\alpha=-\frac{5}{13}$. Moreover, α is in quadrant 3. This means, we can draw the following triangle.

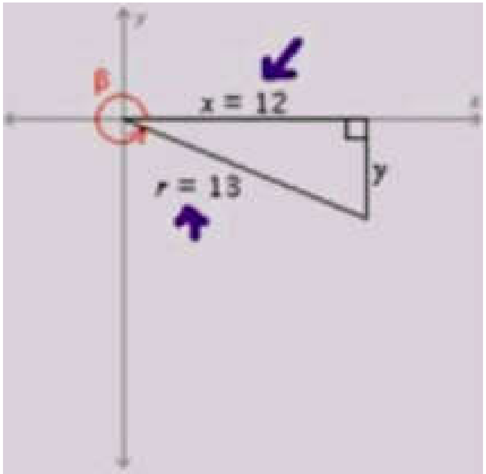


Because, remember, sin(α) is $^y/_r$, and notice we are putting –y, because y is negative in quadrant 3, and *r* is always positive, then we can use the Pythagorean Theorem to find *x* here. Namely:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + (-5)^2 &= 13^2 \\ x^2 + 25 &= 169 \\ x^2 &= 144 \\ x &= \sqrt{144} \\ &= \underline{\pm 12} \end{aligned}$$

Now remember that x is negative in quadrant 3, which means we are going to choose the -possibility here. Remember that we need to know what cos(α) is, and remember that cos(α) = $^x/_r$, which = $^{12}/_{13}$.

It still remains to find the sin(β).Let us do this same thing. We are given that cos(β) = $^{12}/_{13}$, and β is in quadrant 4. Therefore, we can draw the following triangle.



Again, we can use the Pythagorean Theorem to help us find *y*, namely:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (12)^2 + y^2 &= 13^2 \\ y^2 + 144 &= 169 \\ y^2 &= 25 \\ y &= \sqrt{25} \\ &= \underline{\pm 5} \end{aligned}$$

Now since y is negative in quadrant 4, we are going to choose the negative. Remember, we need to find sin(β). Therefore, let us compute that:

$$\begin{aligned} \sin\beta &= \frac{y}{r} \\ &= \underline{\frac{-5}{13}} \end{aligned}$$

Okay, now we have all the pieces to compute cos(α-β).

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \left(\frac{-12}{13}\right)\left(\frac{12}{13}\right) + \left(\frac{-5}{13}\right)\left(\frac{-5}{13}\right) \\ &= \frac{-144}{169} + \frac{25}{169} \\ &= \underline{-\frac{119}{169}} \end{aligned}$$

3 Verifying Trigonometric Identities – Sum & Differences

For example let us verify this identity here.

$$\cos\left(x-\frac{3\pi}{2}\right)=-\sin x$$

Now we see on the left that we have cosine of a difference. Therefore, let us recall the difference formula for cosine.

The difference formula is:

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

We will use that here.

$$\begin{aligned}\cos\left(x-\frac{3\pi}{2}\right)&=\cos x\cos\frac{3\pi}{2}+\sin x\sin\frac{3\pi}{2}\\&=\cos x(0)+\sin x(-1)\\&=\underline{-\sin x}\end{aligned}$$

Therefore, we are left with –sin(x), which, looking up here, is the right-hand side of this identity that we are trying to prove, which means we have verified this identity.

Let us look at another example. Let us verify this identity here.

$$\frac{\sin(x-y)}{\cos x\cos y}=\tan x-\tan y$$

Looking on the left-hand side of this equation, we have the sine of a difference. Therefore, let us recall the difference formula for sine.

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

Using this here on the left-hand side gives us:

$$\begin{aligned}\cos\left(x-\frac{3\pi}{2}\right)&=\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}\\&=\frac{\sin x\cos y}{\cos x\cos y}-\frac{\cos x\sin y}{\cos x\cos y}\\&=\frac{\sin x}{\cos x}-\frac{\sin y}{\cos y}\\&=\underline{\tan x-\tan y}\end{aligned}$$

Moreover, this is exactly the right-hand side of this identity that we wanted to verify.

4 Verifying Trigonometric Identities – Double-Angle Formulas

For example, let us verify this identity.

$$(\sin x+\cos x)^2=1+\sin 2x$$

Remember, we have the following Pythagorean Identity:

$$\sin^2 x+\cos^2 x=1$$

In addition, remember the double-angle formula for sine:

$$\sin 2x=2\sin x\cos x$$

Well, we can start working with the left-hand side:

$$\begin{aligned}(\sin x + \cos x)^2 &= (\sin x + \cos x)(\sin x + \cos x)\\ &= \sin^2 x + \sin x \cos x + \cos x \sin x + \cos^2 x\\ &= \sin^2 x + 2\sin x \cos x + \cos^2 x\\ &= \sin^2 x + \cos^2 x + 2\sin x \cos x\\ &= 1 + 2\sin x \cos x\\ &\underline{= 1 + \sin 2x}\end{aligned}$$

Therefore, we verified this identity.

Okay, let us see another one. Let us verify this identity.

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

Remember, we have the following Pythagorean Identity:

$$1 + \tan^2 x = \sec^2 x$$

Remember the double-angle formulas:

$$\begin{aligned}\sin 2x &= 2\sin x \cos x\\ \cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

Unlike sine and tangent, cosine has three different double-angle formulas, but if we look at them, what we just got is equal to the right-hand side of the first one.

Therefore, we are going to start working here on the right-hand side.

$$\begin{aligned}\cos 2x &= \frac{1-\tan^2 x}{1+\tan^2 x}\\ &= \frac{1-\tan^2 x}{\sec^2 x}\\ &= \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}\\ &= \cos^2 x - \frac{\frac{\sin 2x}{\cos^2 x}}{\frac{1}{\cos^2 x}}\\ &= \cos^2 x - \sin^2 x\\ &\underline{= \cos 2x}\end{aligned}$$

5 Half-Angle Identities

For example, let us use a half-angle identity to find the exact value of:

$$\sin(22.5^\circ)$$

Well, let us recall these identities.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}} = \frac{\sin u}{1+\cos u} = \frac{1-\cos u}{\sin u}$$

Since we are looking for a sin-value, we will use this first identity.

$$\frac{u}{2} = 22.5^\circ$$

$$\underline{u = 45^\circ}$$

That is what we needed to know, therefore, we can plug that value of u into our identity. Now there is one other issue with the formula! What does this + or - mean? Does that mean there are two answers every time?

No, we choose + or - depending upon what quadrant $^u/_2$ lies. Since 22.5°, or $^u/_2$, is in quadrant 1, we will choose the +, because sin-values are + in quadrant 1.

Now we are ready to compute our value.

$$\begin{aligned}
\sin(22.5^\circ) &= \sqrt{1-\cos(45^\circ)} \\
&= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2-\sqrt{2}}{4}} \\
&= \underline{\frac{\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}$$

Alright, let us look at another example. Let us use the half-angle identities to find the exact values of:

$$\cos\frac{5\pi}{12}$$

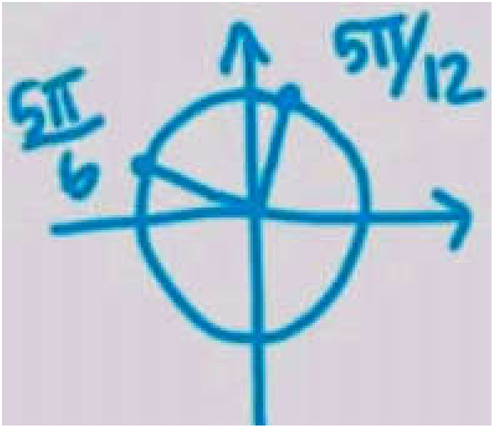
$$\tan\frac{5\pi}{12}$$

Since we are looking for a cosine and a tangent-value, we will be using these last 2 identities. Again, let us:

$$\begin{aligned}
\frac{u}{2} &= \frac{5\pi}{12} \\
u &= 2\left(\frac{5\pi}{12}\right) \\
&= \underline{\frac{5\pi}{6}}
\end{aligned}$$

Notice with the tangent identities, there are three different identities you can use, but we will be using the first one. Again we are going to need to determine whether we are going to choose the + or the - for both of these. Remember what determines that is what quadrant $^u/_2$ lies. Well, since $\frac{5\pi}{12}$ lies in quadrant 1, we will choose the + for both, because both, cosine and tangent, are + in quadrant 1.

Now there is a common mistake that students make often. We want to be sure that we are looking at what quadrant $^u/_2$ lies in, not *u*, to determine whether we are going to choose the + or the -. For example, let us look at our situation here.



Now, since cosine and tangent are both - in quadrant 2, students are tempted to choose the -choices here, because they are thinking about what quadrant *u* lies, rather than what quadrant $^u/_2$ lies.

Do not make that mistake. Always look at what quadrant this $^u/_2$ lies, and not *u*, to determine the + or - in these identities. Alright, let us compute our values.

$$\cos\frac{5\pi}{12} = \sqrt{\frac{1+\cos\frac{5\pi}{6}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\tan\frac{5\pi}{12} = \sqrt{\frac{1-\cos\left(\frac{5\pi}{6}\right)}{1+\cos\left(\frac{5\pi}{6}\right)}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{\frac{2-\sqrt{3}}{2}}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$$

Just be careful that you are looking at what quadrant $^u/_2$ lies, and not *u*, when determining the + or - in these formulas.

Let us look at another example.

$$\sin\theta = -\frac{3}{5} \qquad \textit{and} \qquad \frac{3\pi}{2} < \theta < 2\pi$$

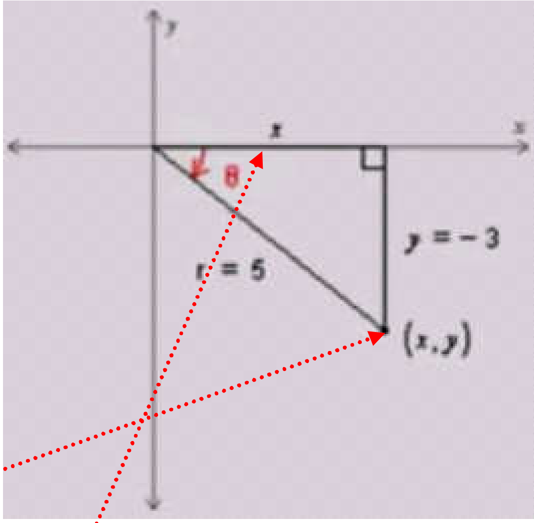
$$\sin\frac{\theta}{2} = ?$$

Let us recall the half-angle identity for sine.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{5}$$

Therefore, we can draw the following triangle



Moreover, if this is $-{}^{3}/_{5}$, notice that we put the - with the *y*, because y is negative in quadrant 4, and *r* is always +. Remember that:

$$\cos\theta = \frac{x}{r}$$

Therefore, if we can find *x* here, then will be able to determine what $\cos(\theta)$ is, which we then can use to determine $\sin\frac{\theta}{2}$. To find *x* we can use the Pythagorean Theorem to help us.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + (-3)^2 &= 5^2 \\ x^2 + 9 &= 25 \\ x^2 &= 16 \\ \underline{x} &\underline{= \pm 4} \end{aligned}$$

However, *x* is in quadrant 4, $x > 0$. Therefore, were going to choose the +value here:

$$\begin{aligned} \sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ \cos\theta &= \frac{x}{r} = \frac{4}{5} \end{aligned}$$

Nevertheless, there is one more issue with this formula, is it not? What does the + or - mean? Does that mean that there are two solutions here? No, we are going to choose either the + or the -, depending upon the quadrant that ${}^{\theta}/_{2}$ lies. That is, if $\sin{}^{\theta}/_{2}$ in that quadrant is +, we will choose the +, and if it is -, we will choose the -.

Now remember, that we are given that:

$$\begin{aligned} \frac{3\pi}{2} &< \theta < 2\pi \\ \frac{3\pi}{4} &< \frac{\theta}{2} < \pi \\ &\underline{\sin\frac{\theta}{2} > 0} \end{aligned}$$

Instead the sines are positive in quadrant 2, $\sin({}^{\theta}/_{2}) > 0$. Therefore, we are going to choose the +value up here in our formula.

Now, there is a common mistake that students make that should be pointed out here. Looking over our figure we see that θ is in quadrant 4, and students will think, because θ is in quadrant 4, and sines are negative in quadrant 4, that they should choose the -value here. Do not look at where θ lies, look at where ${}^{\theta}/_{2}$ lies.

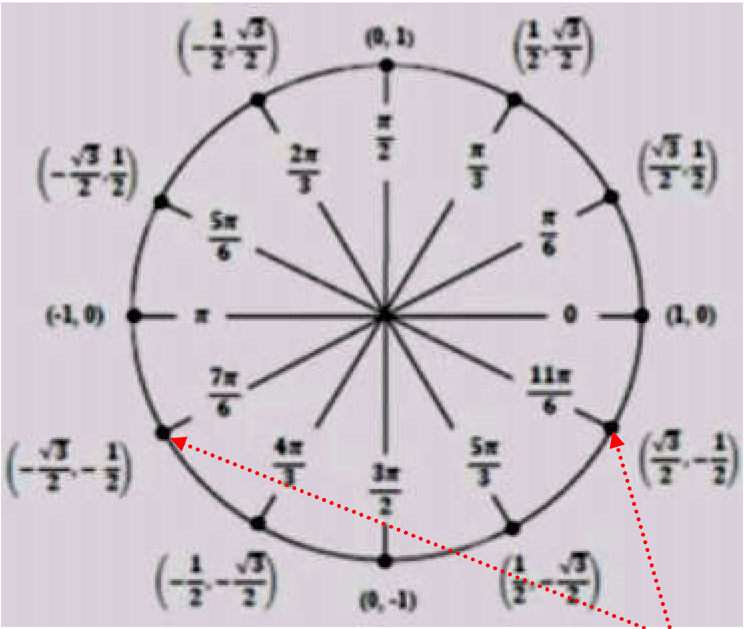
Trigonometric Equations: Law of Sines, Law of Cosines

1 Solving Trigonometric Equations

For example, let us solve this equation:

$$2\sin\theta + 3 = 2 \qquad [0,2\pi)$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$

We need to find all angles θ in the interval $[0,2\pi)$ that have sin = $-{}^1/_2$. Well, let us recall our unit circle.



Remember that the sine is the y-coordinate of the point of intersection of the terminal side of the angle and the unit circle. Therefore, looking down in Quadrant 3 we see that the y-coordinate of this point is $-{}^1/_2$, which corresponds to the angle of ${}^{7\pi}/_6$. Also on Quadrant 4 the y-coordinate of this point is also $-{}^1/_2$, which corresponds to the angle of ${}^{11\pi}/_6$.

Therefore, the sign of both of these angles will equal $-{}^1/_2$. This means, our answer here then is:

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

All right, let us look at another example. Again, let us solve this equation:

$$\sec\theta + 3 = -2 \qquad [0,2\pi)$$
$$\sin\theta = \frac{1}{\cos\theta}$$
$$\cos\theta = \frac{1}{\sec\theta}$$
$$\cos\theta = -\frac{1}{2}$$

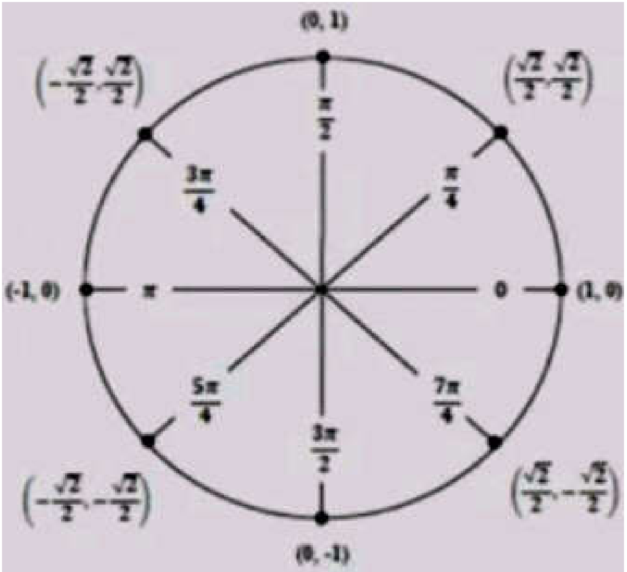
Again, we are calling our unit circle. Moreover, remembering the cosine of θ is the x-coordinate of the point of intersection of the terminal side of the angle and the unit circle. We see that the x-coordinate of the first point in Quadrant 2 is $-{}^1/_2$, which corresponds to the angle of ${}^{2\pi}/_3$. In addition, the x-coordinate of the second point down in Quadrant3 is also $-{}^1/_2$, which corresponds to the angle of ${}^{4\pi}/_3$. Therefore, our answer here would be:

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Let us look at one more example.

$$\cot\theta = 1 \qquad [0,2\pi)$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$= \frac{x}{y}$$

We are looking for angles θ in the interval $[0,2\pi)$ that have the ratio of $^x/_y = 1$, and this will happen at $^\pi/_4$ as well as $^{5\pi}/_4$, because remember our unit circle.



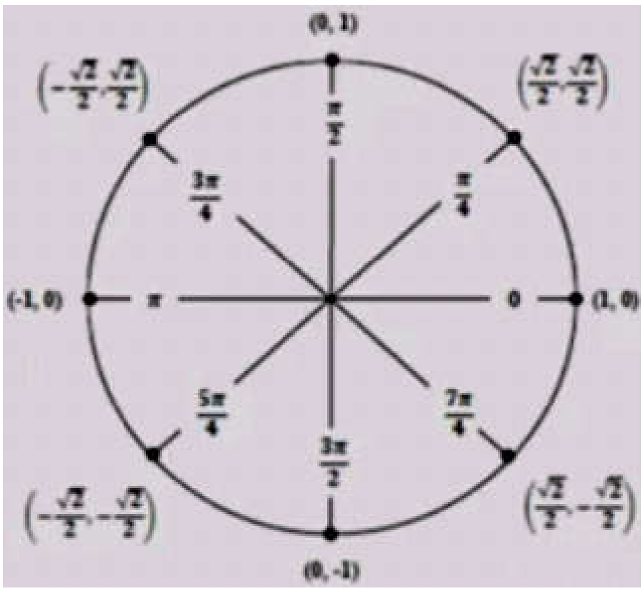
At $^\pi/_4$ the ratio of $^x/_y = 1$, but also down in quadrant 3 the ratio of $^x/_y = 1$, which corresponds to the angle of $^{5\pi}/_4$. In addition, do not forget this angle down in quadrant 3. Most students will only remember the angle in quadrant 1, but the ratio of $^x/_y$ in quadrant 3 will also yield a positive number. Therefore, our answer here is:

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

For example, let us find all solutions to this equation here.

$$2\sin\theta + \sqrt{2} = 0$$
$$2\sin\theta = -\sqrt{2}$$
$$\sin\theta = -\frac{\sqrt{2}}{2}$$

Now remember with your unit circle.



The sine of θ is the Y-coordinate of the point of intersection of the terminal side of the angle and the unit circle. In quadrant 3 we see that the y-coordinate of this point is $-\frac{\sqrt{2}}{2}$, which corresponds to the angle of $5\pi/4$, and also in quadrant 4, the y-coordinate is $-\frac{\sqrt{2}}{2}$, which corresponds to the angle of $7\pi/4$, which means both of these angles will have sine equal to $-\frac{\sqrt{2}}{2}$. Therefore, θ is equal to $5\pi/4$ and $7\pi/4$.

However, what else are these the only two angles that have $\sin = -\frac{\sqrt{2}}{2}$? And they are not, because if we go around this unit circle 2π radians in either directions, we will land on these same two points, which means that the sign of those angles will also $= -\frac{\sqrt{2}}{2}$. Those angles are called terminal angles, because they share the same terminal side.

For example:

$$\frac{5\pi}{4} + 2\pi \qquad \text{co - terminal} \ \frac{5\pi}{4} + 4\pi$$
$$\frac{5\pi}{4} - 2\pi \qquad \text{co - terminal} \ \frac{5\pi}{4} - 4\pi$$

and so on.

All of these angles will have $\sin = -\frac{\sqrt{2}}{2}$. The same thing with the co-terminal angles of $7\pi 4$. All of those angles will also have $\sin = -\frac{\sqrt{2}}{2}$. Not only will these two angles have $\sin = -\frac{\sqrt{2}}{2}$, but also any of there co-terminal angles, and we write that in the following way:

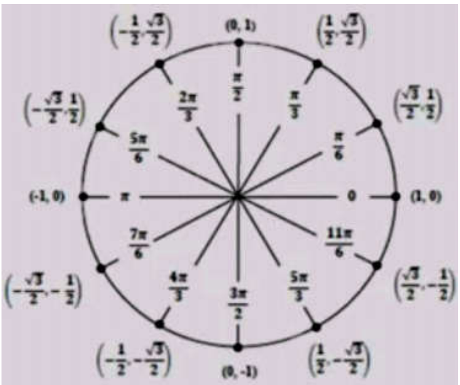
$$\theta = \frac{5\pi}{4} + 2K\pi, K \in Z$$
$$\theta = \frac{7\pi}{4} + 2K\pi, K \in Z$$
$$K = \text{integer}$$

For example, when K = 0 we get the angle of $5\pi/4$ and when K = 1 we get the angle of $7\pi/4$.

All right, let us look at another example. Let us find all solutions to this equation.

$$2\cos^2\theta + \cos\theta = 0$$
$$\cos\theta(2\cos\theta + 1) = 0$$
$$\cos\theta = 0 \qquad or \qquad 2\cos\theta + 1 = 0$$

Again, thinking of our unit circle:



Remember, that the cosine of θ is the x-coordinate of the point of intersection of the terminal side of the angle and the unit circle. We see that the x-coordinate of this point is = 0. This corresponds to the angle of $\pi/2$. Also down there, the x-coordinate is also 0. This corresponds to the angle of $3\pi/2$, and so for the first equation the solution would be:

$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

or any of their co-terminal angles, which, remember from the last example, we can write:

$$\theta = \frac{\pi}{2} + 2K\pi, K \in Z$$
$$\theta = \frac{3\pi}{2} + 2K\pi, K \in Z$$

Now looking back at the unit circle in this type of situation, where our two solutions are exactly π units away from each other. We do not necessarily have to write both of these statements over here. We can consolidate them into one statement, realizing that we only need to add or subtract integer multiples of π from $\pi/2$ to get to another solution to this equation. That is we can condemn these two statements by writing:

$$\theta = \frac{\pi}{2} + K\pi, K \in Z$$

This is the solutions to the first equation. Now if you had left both of the statements above, you would have been correct. This is just a more condensed way of writing the answer.

All right, let us look at the solutions to the second equation.

$$2\cos\theta + 1 = 0$$
$$\cos\theta = -\frac{1}{2}$$

Again, remembering that cos(θ) is the x-coordinate of the point of intersection of the terminal side of the angle on the unit circle. Looking at our unit circle, we see in quadrant 2 that the x-coordinate of the point is -½, which corresponds to the angle $2\pi/3$, but also down there in quadrant 3 the x-coordinate is also -½. This corresponds to this angle $4\pi/3$, which means both of those angles have cos = -½, but so do all of their co-terminal angles. This means the solutions to this second equation are given by:

$$\theta = \frac{2\pi}{3} + 2K\pi, K \in Z$$
$$\theta = \frac{4\pi}{3} + 2K\pi, K \in Z$$

Moreover, since these solutions are not π units away from each other, we cannot consolidate them.

Let us work out solving equations. For example, let us find our solutions to this equation:

$$\tan\theta - \sqrt{3} = 0$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{y}{x}$$
$$= \sqrt{3}\,?$$

Remembering our unit circle, the sin(θ) is the y-coordinate of the point of interception of the terminal side of its angle and the unit circle, and the cos(θ) is the x-coordinate of the point of intersection of the terminal side of its angle and the unit circle, which means, we want to find out where this ratio is $= \sqrt{3}$.

Look in quadrant 1. If we take the ratio of y/x, which corresponds to the angle of $\pi/3$, we will get $\sqrt{3}$, but also if we take the ratio of y/x in quadrant 3, which corresponds to the angle of $4\pi/3$, we also get $\sqrt{3}$. That is:

$$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$
$$\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

Which means that the tangent of both of these angles here will $= \sqrt{3}$, but not only will the tangent of those two angles $= \sqrt{3}$, but the tangent of any of their co-terminal angles will also $= \sqrt{3}$.

$$\tan\theta = \sqrt{3}$$
$$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$$

or any of their co-terminal angles. Which remember from the last example, we can write:

$$\theta = \frac{\pi}{3} + 2K\pi, K \in Z$$
$$\theta = \frac{4\pi}{3} + 2K\pi, K \in Z$$

Remember, when these angles are exactly π units away from each other, then we can consolidate these two statements into one, namely:

$$\theta = \frac{\pi}{3} + K\pi, K \in Z$$

All right, let us look at another example. Let us find all solutions to this equation.

$$\csc\theta = 2$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sin\theta = \frac{1}{\csc\theta}$$
$$\sin\theta = \frac{1}{2}$$

Again, think of our unit circle and remember, that the sin(θ) is the y-coordinate of the point of intersection of its terminal side and the unit circle. Looking above we see that the y-coordinate of one point in quadrant 1 is ½, which corresponds to the angle of $\pi/6$, but also in quadrant 2 the y-coordinate of one point is ½, which corresponds to $5\pi/6$. This means that the sin of both of those angles is = ½, but not only have those two angles, but also any of their co-terminal angles. That is:

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

or any of their co-terminal angles. This is written as:

$$\theta = \frac{\pi}{6} + 2K\pi, K \in Z$$
$$\theta = \frac{5\pi}{6} + 2K\pi, K \in Z$$

1.1 Solving Trigonometric Equations – Quadratic Form

Let us find all solutions to this equations here.

$$2\sin^2\theta + \sin\theta = 1$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$

Which means, either the first factor is zero, or the second factor is zero. Let us solve each equation separately. We will start with this first equation.

$$2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

or any of their co-terminal angles. Remember how we write this.

$$\theta = \frac{\pi}{6} + 2K\pi, K \in Z$$
$$\theta = \frac{5\pi}{6} + 2K\pi, K \in Z$$

All of these angles will be solutions to this first equation. Now, what about this second equation?

$$\sin\theta + 1 = 0$$
$$\sin\theta = -1$$
$$\theta = \frac{3\pi}{2}$$

or any of its co-terminal angles, that is:

$$\theta = \frac{3\pi}{2} + 2K\pi, K \in Z$$

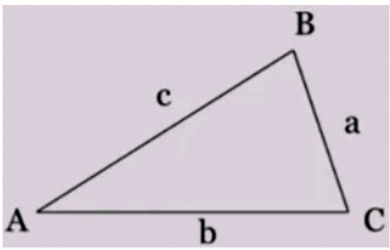
Any one of these values of θ will satisfy that original equation.

2 Law of Sines and Cosines – ASA-Case

Let triangle *A*, *B*, *C* be an oblique triangle with A = 41°, B = 70° and c = 5. Let us solve the triangle.

An oblique triangle is one that is not a right triangle, and we use the law of sines and the law of cosines to solve oblique triangles, or to find all the missing angles and sides. However, we must be given three of the six parts of the triangle, with at least one side given, which is *c*, therefore, we can solve this triangle.

Now, when working with solving oblique triangles by using the law of sines and cosines, we label the side across from each angle with the same letter, however, just lower case. In other words:



We are given that A = 41°, B = 70° and the side between them, *c*, has the length 5. Now, we are in what is called the ASA-case, which stands for Angle-Side-Angle, because we are given 2 angles and the side between them, or the included side.

When we are in this case, we can use the law of sines to help us, which states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

However, in order to apply this, we need a matching angle inside. Looking over in our figure, because we are given *c*, let us find *C*. We can do that by using the fact that the angle measures in a triangle add up to 180°, which means:

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &= 180^\circ - (41^\circ + 70^\circ) \\ &= \underline{69^\circ} \end{aligned}$$

Now, looking in our triangle, we have a matching pair. We know *C* and we know *c*. Let us find *a* and *b*. We will begin with *a* and use the fact that:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin C}{c} \\ \frac{\sin(41^\circ)}{a} &= \frac{\sin(69^\circ)}{5} \\ a * \sin(69^\circ) &= 5 * \sin(41^\circ) \\ a &= \frac{5 * \sin(41^\circ)}{\sin(69^\circ)} \\ &\approx \underline{3.51} \end{aligned}$$

Now it remains to find *b*, which we can do by using the fact that:

$$
\begin{aligned}
\frac{\sin B}{b} &= \frac{\sin C}{c} \\
\frac{\sin(70^\circ)}{b} &= \frac{\sin(69^\circ)}{5} \\
b * \sin(69^\circ) &= 5 * \sin(70^\circ) \\
b &= \frac{5 * \sin(70^\circ)}{\sin(69^\circ)} \\
&\approx \underline{5.03}
\end{aligned}
$$

3 Law of Sines and Cosines – SAS-Case

For example, let triangle A B C be an oblique triangle with A = 111°, b = 74 and c = 29. Let us solve this triangle.

This is what we call the SAS-case, which stands for Side-Angle-Side, because we are given two sides and the angle between them. When we are in this case we can use the law of cosines to help us, and here are the forms.

Law of Cosines:

Suppose a triangle has angles A, B, and C with opposite sides of a, b, and c, respectively.
Then, the law of cosines says the following.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}
$$

Notice that the side that we are squaring on the left is the side that corresponds to this angle on the right. Also the two sides that we are squaring on the right uniting together are the same two sides that we are multiplying by 2, and then by the cosine of *a*, because we are given *A*, *b* and *c* we will be using this first equation. We can solve for a as follows:

$$
\begin{aligned}
a^2 &= 74^2 + 29^2 - 2 * 74 * 29 * \cos(111^\circ) \\
a &= \sqrt{74^2 + 29^2 - 2 * 74 * 29 * \cos(111^\circ)} \\
&\approx \underline{88.6291}
\end{aligned}
$$

Now we need to find *B* and *C*. Both of which are acute, right? Because there can be at most one obtuse angle in a triangle. Let us start by solving for *B*, which we can do by using the law of sines.

$$
\begin{aligned}
\frac{\sin B}{b} &= \frac{\sin A}{a} \\
\sin B &= \frac{b \sin A}{a} \\
&\approx \frac{74 * \sin(111^\circ)}{88.6291} \\
B &= \sin-1 * \left(\frac{74 * \sin(111^\circ)}{88.6291}\right) \\
&\approx \underline{51.2134^\circ}
\end{aligned}
$$

Notice that we did not need to consider the other angle that is less than 180°, whose sign is:

$$
\begin{aligned}
C &= 180^\circ - (A + B) \\
&\approx 180^\circ - (111^\circ + 51.2134^\circ) \\
&\approx \underline{17.7866^\circ}
\end{aligned}
$$

4 Law of Sines and Cosines – SSA--Case

For example, let triangle A B C be an oblique triangle with b = 37, a = 54 and B = 30°. Let us solve the triangle.

This is what we call the SSA-case, which stands for Side-Side-Angle, because we are given two sides and an angle that is not included between them.

Since we are given a matching angle and a side pair, namely *B* and *b*, we can apply the law of sines, and since we are given *a*, we can use the law of sines to find *A*. Namely:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \sin A &= \frac{a \sin B}{b} \\ &= \frac{54 * \sin(30^\circ)}{37} \\ &= \frac{54 * \left(\frac{1}{2}\right)}{37} \\ &= \frac{27}{37} \\ &\approx \underline{0.7297} \end{aligned}$$

Since *A* is an angle in a triangle, it degree measure must be between 0° and 180°, and therefore, there are two possibilities for *A*, let us call them A_1 and A_2.

$$\begin{aligned} A_1 &= \sin^{-1}\left(\frac{27}{37}\right) \approx 46.8612^\circ \\ A_2 &= 180^\circ - \sin^{-1}\left(\frac{27}{37}\right) \approx 133.1388^\circ \end{aligned}$$

Now the question is, do they both work? Well, they will both work as long as, when we add *B* to these measures, we are still < 180°.

$$\begin{aligned} A1 + B &\approx 46.8612^\circ + 30^\circ \\ &\approx 76.8612^\circ < 180^\circ \\ A2 + B &\approx 133.1388^\circ + 30^\circ \\ &\approx 163.1388^\circ < 180^\circ \end{aligned}$$

That means that we have two solutions. We can form two triangles with this given information. If we know *A* and *B*, then we can find *C*, because the angles measure in a triangle have to add up to 180°. That is:

$$\begin{aligned} \text{First solution}: A &= A_1 \approx 46.8612^\circ \\ C &= 180^\circ - (A + B) \\ &\approx 180^\circ - (46.8612^\circ + 30^\circ) \\ &\approx \underline{103.1388^\circ} \\ \frac{c}{\sin C} &= \frac{b}{\sin B} \\ c &= \frac{b \sin C}{\sin B} \\ &\approx \frac{37 * \sin(103.1388^\circ)}{\sin(30^\circ)} \\ &\approx \underline{72.0629} \end{aligned}$$

$$\begin{aligned} \text{Second solution}: A &= A_2 \approx 133.1388^\circ \\ C &= 180^\circ - (A + B) \\ &\approx 180^\circ - (133.1388^\circ + 30^\circ) \\ &\approx \underline{16.8612^\circ} \\ \frac{c}{\sin C} &= \frac{b}{\sin B} \\ c &= \frac{b \sin C}{\sin B} \\ &\approx \frac{37 * \sin(16.8612^\circ)}{\sin(30^\circ)} \\ &\approx \underline{21.4641} \end{aligned}$$

Let us round our answers to the nearest tenth, and there are two solutions.

$$\begin{array}{ll} \text{First solution}: & \text{Second solution}: \\ A = 46.9^\circ & A = 133.1^\circ \\ C = 103.1^\circ & C = 16.9^\circ \\ c = 72.1 & c = 21.5 \end{array}$$

5 Law of Sines and Cosines – SSS-Case

For example, let triangle A B C be an oblique triangle with a = 10, b = 14, and c = 6. Let us solve the triangle.

Now this is what we refer to as the SSS-case. This stands for Side-Side-Side, because we are given all three sides of the triangle, but no angles. Therefore, since we are not given a matching angle-side-pair, we cannot start off by using the law of sines, but, however, we can start off by using the law of cosines.

When we are in the SSS-case we are going to be using the law of cosines. What we do first is we find the largest angle to see if our triangle has any obtuse angles. Therefore, we will find the largest angle first, and we know what angle that is, because it is across from the longest side. Since b is the longest side, which means B is the largest angle, we will be using this middle formula down here.

Law of Cosines:

$$\begin{aligned} a^2 &= b^2+c^2-2bc\cos A \\ b^2 &= a^2+c^2-2ac\cos B \\ c^2 &= a^2+b^2-2ab\cos C \end{aligned}$$

That is:

$$\begin{aligned} b^2 &= a^2+c^2-2ac*\cos B \\ 14^2 &= 10^2+6^2-2*10*6*\cos B \\ \cos B &= \frac{10^2+6^2-14^2}{2*10*6} \\ &= -\frac{1}{2} \\ B &= \cos^{-1}\left(-\frac{1}{2}\right) \\ &\underline{=120^\circ} \end{aligned}$$

Now we have a matching angle side pair, B, and b. Let us use the law of sines to find A.

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \sin A &= \frac{a\sin B}{b} \\ &= \frac{10\sin(120^\circ)}{14} \\ &= \frac{10*\frac{\sqrt{3}}{2}}{14} \\ &= \frac{5\sqrt{3}}{14} \\ A &= \sin^{-1}\left(\frac{5\sqrt{3}}{14}\right) \\ &\underline{\approx 38.2132^\circ} \end{aligned}$$

There is another angle A that is < 180°, whose sine is equal to this $\frac{5\sqrt{3}}{14}$, and therefore, can be a candidate for an angle in a triangle. Namely:

$$180^\circ - 38.2132 = 141.7868$$

However, that angle would be obtuse, and since *B* is already obtuse, there is no way that obtuse possibility would work for *A*, because, remember, there can be at most one obtuse angle in a triangle. Therefore, *A* is approximately 38.2132°.

Now it still remains to find *C*, but we can do so by using the fact that the angles measured in a triangle add up to 180°. That is:

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &\approx 180^\circ - (38.2132^\circ + 120^\circ) \\ &\approx \underline{21.7868^\circ} \end{aligned}$$

Now looking over here, you might be wondering why we started off by finding the largest angle first. Let us say that, instead of solving for the largest angle *B* first, we decided to solve for *A* first. Therefore, we would be using this first formula of the law of cosines, and if we solve this for cos(A) we would get:

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{14^2 + 6^2 - 10^2}{2 * 14 * 6} \\ &= \frac{11}{14} \\ A &= \cos - 1\left(\frac{11}{14}\right) \\ &\approx \underline{38.2132^\circ} \end{aligned}$$

This is the same answer that we found. Now, if we use the law of sines to try to find *B*, we have that:

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin A}{a} \\ \sin B &= \frac{b \sin A}{a} \\ &= \frac{14 \sin\left(\cos^{-1}\left(\frac{11}{14}\right)\right)}{10} \\ B &= \sin - 1\left(\frac{14 \sin\left(\cos - 1\left(\frac{11}{14}\right)\right)}{10}\right) \\ &= \underline{60^\circ} \end{aligned}$$

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &\approx \underline{81.7868^\circ} \end{aligned}$$

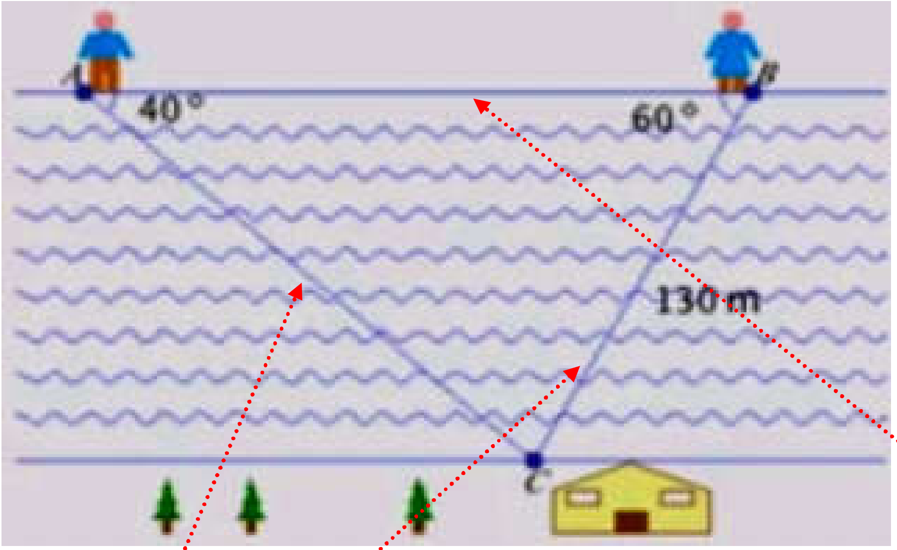
which is wrong, because how can *C* be larger then *B*? Looking at our triangle, *c* is smaller than *b*, and, therefore, *C* has to be smaller than *B*.

What students forget, looking back over here on the right, this angle of 60° is not the only angle whose sine is equal to this quantity here, is it? There is also 180°-120°, but by solving for the largest angle first, and assessing whether or not there is an obtuse angle in the triangle, then we know that the other two angles have to be acute. We would not have to worry about the other possibility for *b*. Therefore, if you always solve for the largest angle first, then the other two have to be acute. Be careful with the SSS-case.

6 Law of Sines Word Problems

For example, Jacob and Natalie are standing on a river bank at points *A* and *B* respectively. Natalie is 130 m from a house located across the river at point *C*. Suppose the angle *A* is 40° and the angle B is 60°. How far are Jacob and Natalie standing from one another?

The situation is shown here in this figure.



Remember we label the sides opposite the angles with the same letter, but just lowercase. Since we are looking for how far apart Jacob and Natalie are standing from one another, which means we are looking for *c*, which is across from *C*, and this is *b* and this is *a*.

Since we want to find *c*, we need to know *C* in order to apply the law of sines. However, since the sum of the angle measures in a triangle is 180°, we have the following:

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &= 180^\circ - (40^\circ + 60^\circ) \\ &= \underline{80^\circ} \end{aligned}$$

Now we know *A*, *a*, and *C*. This means we can find *c* by using the law of sines, namely:

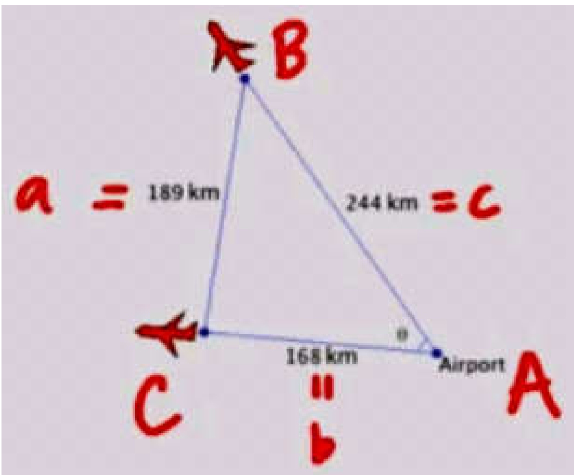
$$\begin{aligned} \frac{\sin C}{c} &= \frac{\sin A}{a} \\ c &= \frac{a * \sin(C)}{\sin(A)} \\ &= \frac{130 * \sin(80^\circ)}{\sin(40^\circ)} [m] \\ &\approx \underline{199.2m} \end{aligned}$$

Therefore, Jacob and Natalie are standing approximately 199.2 m apart.

7 Law of Cosines Word Problems

Two airplanes leave the airport at the same time. 1 h later they are 189 km apart. If one plane traveled 168 km and the other plane traveled 244 km during that hour, find the angle θ between their flight paths.

Now the figure here shows the situation.



We are looking for angle θ, which is *A*. Therefore, the following law of cosine formula can help us.

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc * \cos A \\
189^2 &= 168^2 + 244^2 - 2 * 168 * 244 * \cos A \\
\cos A &= \frac{168^2 + 244^2 - 189^2}{2 * 168 * 244} \\
A &= \cos^{-1}\left(\frac{168^2 + 244^2 - 189^2}{2 * 168 * 244}\right) \\
&\approx \underline{50.6^\circ}
\end{aligned}$$

Looking back up here in our figure then, remember *A* is θ, which is what we were looking for, therefore, $\theta \approx 50.6°$.